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# SOME PROPERTIES OF THE WEIGHTED LINDLEY DISTRIBUTION

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## ABSTRACT

*A one parameter Lindley distribution is considered in this paper. The main aim of this paper is to develop weighted Lindley distribution. To derive the weighted Lindley distribution, a new weight function has been used in this paper. The moments and related properties of the weighted Lindley distribution have been established. Moreover its survival function, cumulative hazard rate function, failure rate function, mean residual life function and entropy have been discussed. The stochastic ordering of the weighted Lindley distribution is developed. The trend of the hazard rate function of the weighted Lindley distribution is discussed through a lemma.*

**KEY WORDS:** *Weighted distribution; WLD; stochastic ordering; moments; pdf; cdf; mean residual life; entropy; hazard rate function.*

## 1.INTRODUCTION

Lindley (1958) introduced a single parameter distribution named Lindley distribution having probability distribution function (pdf)

$$f(x) = \frac{\theta^{2}}{(\theta+1)}(1+x)e^{-\theta x}, \qquad x>0, \theta>0 \qquad (1.1)$$

The Lindley distribution is the mixture of exponential (θ) and Gamma (2, θ) distributions. Sankaran (1970) introduced the Poisson-Lindley distribution to model count data and the distribution arises from the Poisson and Lindley distributions. Ghitany et al (2008) discussed various properties of the Lindley distribution with applications and showed that sometimes Lindley distribution better fits than exponential distribution. Ghitany and Al-Mutairi (2008) introduced the size-biased Poisson-Lindley distribution with its applications to some real data set. Shanker and Mishra (2013) developed the quasi Lindley distribution (QLD) of which Lindley is a particular case. They derived the moments, hazard rate function. Mean residual life and stochastic ordering of the quasi Lindley distribution and proved that QLD provides closer fit than Lindley distribution. Shanker et al (2013) introduced a two parameter Lindley distribution and derived its moments, hazard rate function, mean residual life and stochastic ordering. They apply the distribution on some data sets related to waiting times and survival times. Adhikari and Srivastava (2014) presented the Poisson-size-biased Lindley distribution and discussed its various propertied. A simulation study is also proposed in this paper. Shanker and Mishra (2014) introduced a two parameter Poisson-Lindley distribution of which Sankaran's (1970) one parameter Poisson-Lindley distirbuion is a particular case. They derived its moments, estimations of parameters and fit the distribution on some data set. Abouammoh et al (2015) proposed a new generalized Lindley distribution mixture of two gamma distributions. They established the size-biased, the length-biased, Lorenze curve, estimation of parameters and fit the model on some real data set.

When a researcher records observations by nature to a certain stochastic model, the observations will not have the original distribution unless every observation has an equal chance of being recorded. Rao (1965) introduced the distributions for this type of problems and named them weighted distributions. The weighted distribution with weight function w(x) is called weighted distribution and is defined as

$$f(x)=\frac{w(x)}{E(w(x))}f_0(x), \qquad w(x)>0. \tag{1.2}$$

If $w(x)=x$, then the weighted distributions are called size-biased distributions. Weighted distribution theory applies where biased data arise. For example recorded observations will be biased and not have the original distribution unless every observation is given an equal chance of being recorded. Patil and Rao (1978) discussed some general models leading to weighted distributions and showed that how the size-biased distributions occurs in natural way in many sampling problems. They consider the certain basic distributions and their size-biased forms. Sunoj and Maya (2006) introduced relationships between weighted and original variables in the context of repairable system. Furthermore they prove some characterizations for some specific models. Patil and Rao (1977) discussed a list of important weight functions which are used in discrete and continuous models.

## 2.WEIGHTED LINDLEY DISTRIBUTION

Using a one parameter Lindley distribution with parameter $\theta$ defined by the probability distribution function (pdf) given in (1.1) and the weight function $w(x)=e^{\phi x}$, we derived the weighted Lindley distribution (WLD) having probability distribution function (pdf)

$$f(x)=\frac{(\theta-\phi)^2}{(\theta-\phi+1)}(1+x)e^{-x(\theta-\phi)},$$
$$x>0,\theta>0,\theta>\phi. \tag{2.1}$$

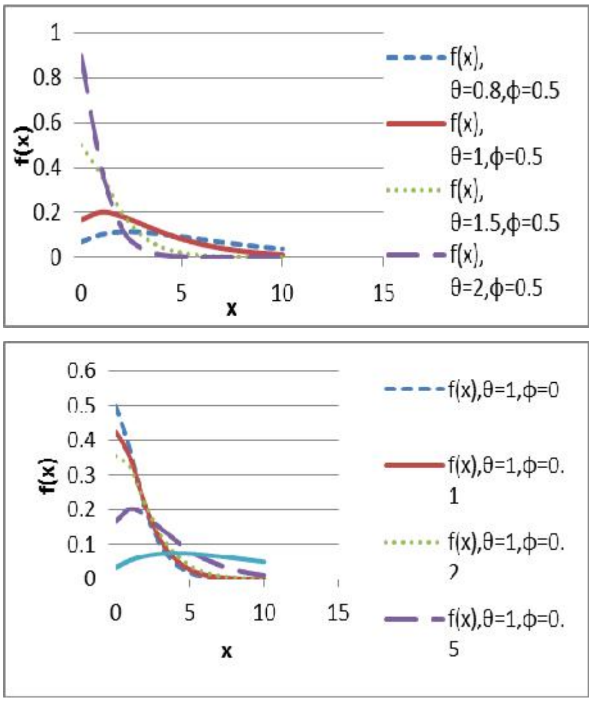



**Fig.2.1 pdf graph of WLD**

The cumulative distribution function (cdf) of the weighted Lindley distribution (WLD) is given by

$$F(x)=1-\frac{(\theta-\phi+1)+(\theta-\phi)x}{(\theta-\phi+1)}e^{-x(\theta-\phi)},$$
$$x>0,\theta>0,\theta>\phi. \tag{2.2}$$

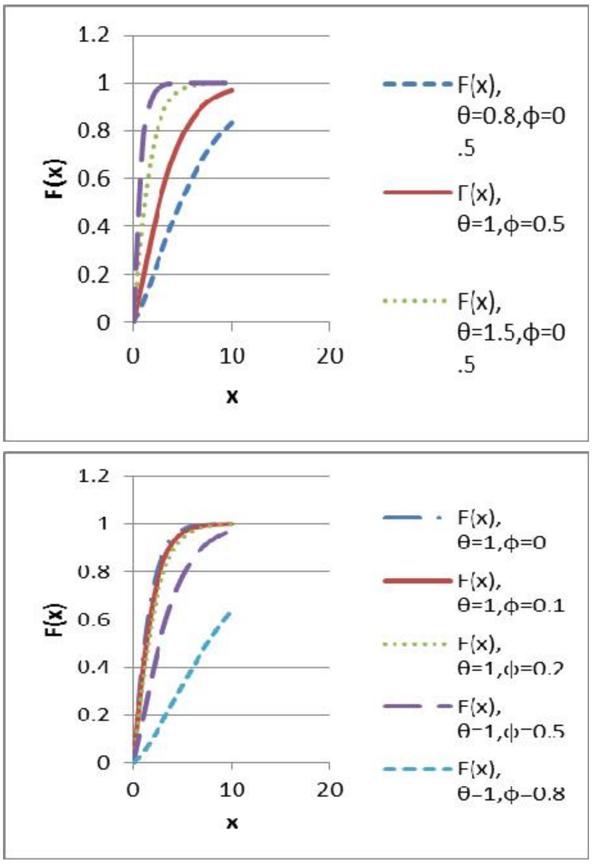



**Fig. 2.2 cdf graphs of the WLD**

## 3. MOMENTS AND RELATED PROPERTIES

The moment generating function of the WLD is

$$M_x(t)=\frac{(\text{¯}-\text{¤})(\text{¯}-\text{¤}-t+1)}{(\text{¯}-\text{¤}+1)(\text{¯}-\text{¤}-t)^2},\quad \text{¯}>\text{¤}. \tag{3.1}$$

The rth moments about origin of the weighted Lindley distribution (WLD) are defined as

$$\text{«}'_r=\frac{(\text{¯}-\text{¤}+r+1)\Gamma(r+1)}{(\text{¯}-\text{¤}+1)(\text{¯}-\text{¤})^r},\quad \text{¯}>\text{¤}. \tag{3.2}$$

For $r=1,2,3,4.$ the first four moments about origin of the WLD are

$$\text{«}'_1=\frac{(\text{¯}-\text{¤}+2)}{(\text{¯}-\text{¤}+1)(\text{¯}-\text{¤})},\quad \text{¯}>\text{¤},$$

$$\text{«}'_2=\frac{2(\text{¯}-\text{¤}+3)}{(\text{¯}-\text{¤}+1)(\text{¯}-\text{¤})^2},\quad \text{¯}>\text{¤}$$

$$\text{«}'_3=\frac{6(\text{¯}-\text{¤}+4)}{(\text{¯}-\text{¤}+1)(\text{¯}-\text{¤})^3},\quad \text{¯}>\text{¤},$$

$$\text{«}_4{}'=\frac{24(\text{¯}-\text{¤}+5)}{(\text{¯}-\text{¤}+1)(\text{¯}-\text{¤})^4},\quad \text{¯}>\text{¤}.$$

The central moments of the WLD are obtained as

$$\text{«}_2=\frac{(\text{¯}-\text{¤})^2+2[2(\text{¯}-\text{¤})+1]}{(\text{¯}-\text{¤}+1)^2(\text{¯}-\text{¤})^2},\quad \text{¯}>\text{¤}. \tag{3.3}$$

$$\text{«}_3=\frac{2\left[2+6(\text{¯}-\text{¤})+6(\text{¯}-\text{¤})^2+(\text{¯}-\text{¤})^3\right]}{(\text{¯}-\text{¤}+1)^3(\text{¯}-\text{¤})^3},\quad \text{¯}>\text{¤}. \tag{3.4}$$

$$\text{«}_4=\frac{3\left[8+40(\text{¯}-\text{¤})-16(\text{¯}-\text{¤})^2-(\text{¯}-\text{¤})^4\right]}{(\text{¯}-\text{¤}+1)^4(\text{¯}-\text{¤})^4},\quad \text{¯}>\text{¤}. \tag{3.5}$$

The coefficient of skewness $(\ _1)$ and kurtosis $(\ _2)$ of the WLD are defined as

$$_1=\frac{4\left[2+6(\text{¯}-\text{¤})+6(\text{¯}-\text{¤})^2+(\text{¯}-\text{¤})^3\right]^2}{(\text{¯}-\text{¤}+1)^3(\text{¯}-\text{¤})^3\left[(\text{¯}-\text{¤})^2+2\{2(\text{¯}-\text{¤})+1\}\right]^3},\quad \text{¯}>\text{¤}. \tag{3.6}$$

$$_2=\frac{3\left[8+40(\text{¯}-\text{¤})-16(\text{¯}-\text{¤})^2-(\text{¯}-\text{¤})^4\right]}{(\text{¯}-\text{¤}+1)^2(\text{¯}-\text{¤})^2\left[(\text{¯}-\text{¤})^2+2\{2(\text{¯}-\text{¤})+1\}\right]^2},\quad \text{¯}>\text{¤}. \tag{3.7}$$

A commonly used measure of variability of a random variable x is the Fisher index of dispersion is defined by

$$FI(X)=\frac{Var(X)}{E(X)}$$

$$FI(X)=\frac{\pm^2}{\text{«}'_1}=\frac{\left[(\text{¯}-\text{¤})^2+2\{2(\text{¯}-\text{¤})+1\}\right]}{(\text{¯}-\text{¤})(\text{¯}-\text{¤}+1)(\text{¯}-\text{¤}+2)},\quad \text{¯}>\text{¤}. \tag{3.8}$$

FI(X) could be greater than or less than one. The distribution is over-dispersed if FI(X) >1, equi-dispersed if FI(X) = 1, and under-dispersed if FI(X) < 1.

The median of the WLD is given as

$$m=-\ln\frac{(\text{¯}-\text{¤}+1)}{2(\text{¯}-\text{¤})[(\text{¯}-\text{¤}+1)+(\text{¯}-\text{¤})m_0]},\quad \text{¯}>\text{¤}. \tag{3.9}$$

The $m_o$ is the initial value of m (median). The equation (3.9) can be solved iteratively till the closer value of m is obtained.

The mode of WLD is defined as

$$\hat{x}_m=\frac{1-\text{¯}+\text{¤}}{\text{¯}-\text{¤}}. \tag{3.10}$$

It can be easily seen that the mean is always greater than the mode of the WLD as

$$\text{mod}\,e<mean$$

$$\frac{1-\text{¯}+\text{¤}}{\text{¯}-\text{¤}}<\frac{\text{¯}-\text{¤}+2}{(\text{¯}-\text{¤}+1)(\text{¯}-\text{¤})}$$

$$\text{¯}^2-\text{¤}^2<\text{¯}-\text{¤}+1.$$

The quantile function of the WLD is given as

$$x=\frac{-\ln[(\text{¯}-\text{¤}+1)(1-P)/\{(\text{¯}-\text{¤})-(\text{¯}-\text{¤})x_0\}]}{(\text{¯}-\text{¤})}. \tag{3.11}$$

$x_o$ is the initial value of x to obtain the closer value of x. P is a probability, at 50%, 25% and 75%, we can get the distribution of median, 1st quartile and 3rd quartile respectively.

## 4. FAILURE RATE, MEAN RESIDUAL LIFE AND ENTROPY

The survival function of the WLD is

$$S(x)=\frac{(¯ - ¤ + 1)+(¯ - ¤)x}{(¯ - ¤ + 1)}e^{-x(¯ - ¤)}, \quad ¯ > ¤. \tag{4.1}$$

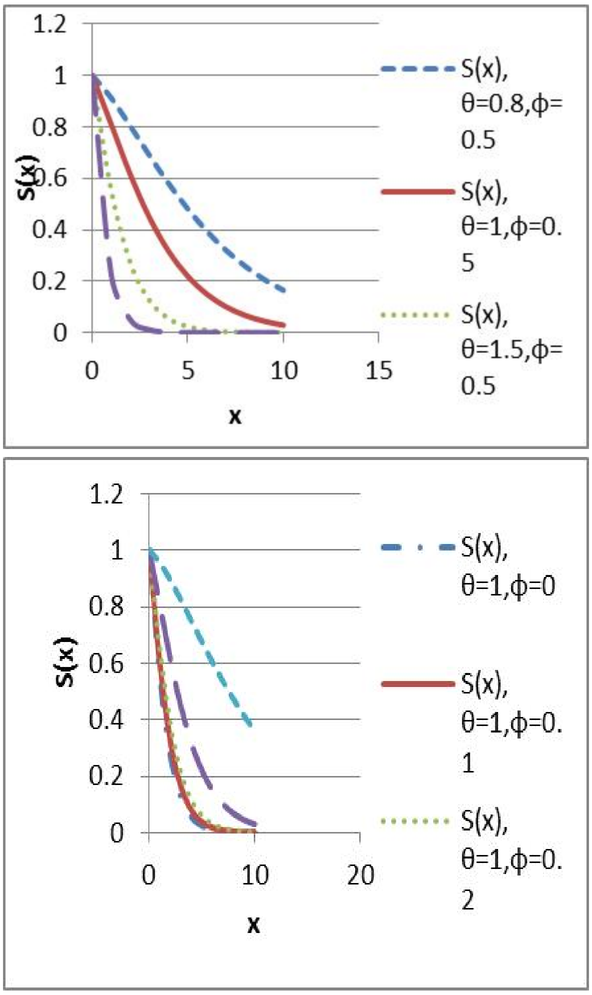



**Fig 4.1 survival graphs of the WLD**

The cumulative hazard rate function of the WLD is

$$H(x)=-\ln\left[\frac{(¯ - ¤ + 1)+(¯ - ¤)x}{(¯ - ¤ + 1)}\right]-x(¯ - ¤) \quad ¯ > ¤. \tag{4.2}$$

The failure rate function of the WLD is

$$h(x)=\frac{(¯ - ¤)^2}{[(¯ - ¤ + 1)+(¯ - ¤)x]}(1+x), \quad ¯ > ¤. \tag{4.3}$$

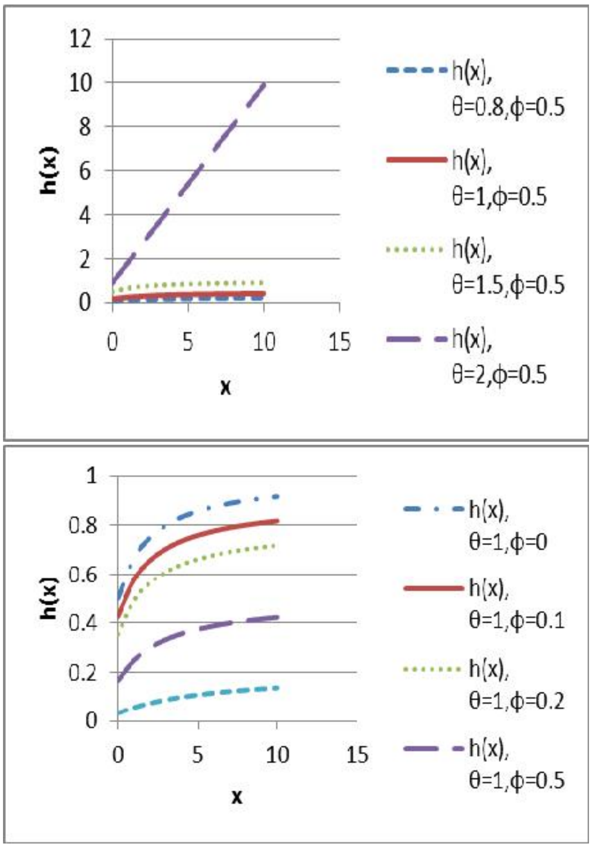
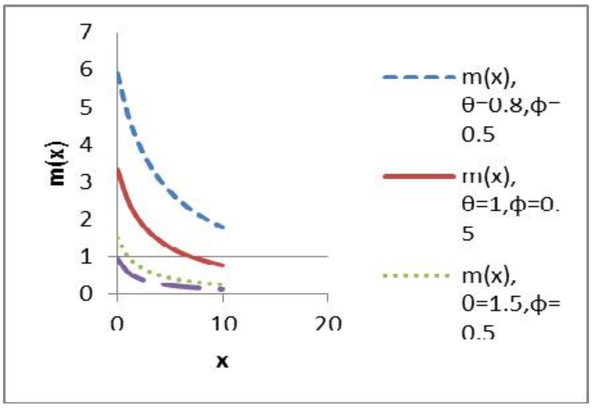



**Fig 4.2 hazard rate function graphs of the WLD**

And the mean residual life of the WLD is given as

$$m(x)=\frac{(¯ - ¤ + 2)}{(¯ - ¤)[(¯ - ¤ + 1)+(¯ - ¤)x]}, \quad ¯ > ¤. \tag{4.4}$$



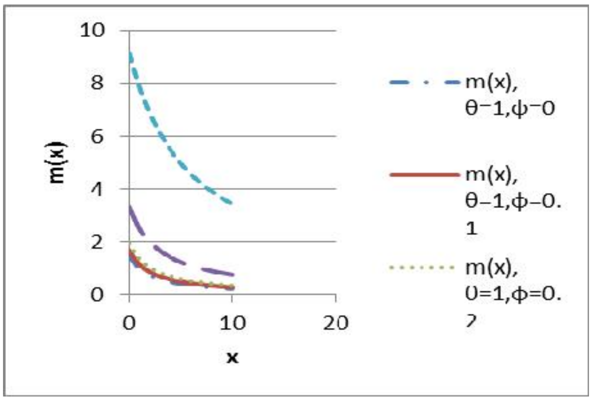



**Fig.4.3 mean residual graphs of the WLD**

From equation (4.3) and (4.4) it can be seen that

$$h(0)=\frac{(¯-¤)^2}{(¯-¤+1)}=f(0) \quad \text{and}$$

$$m(0)=\frac{(¯-¤+2)}{(¯-¤)(¯-¤+1)}=«_1'.$$

The Entropy of the WLD is

$$H(x)=E[I(X)]=E[-\ln f(x)]$$

$$H(x)=\frac{¯-¤+2}{¯-¤+1}-\ln\left[\frac{(¯-¤)^2}{(¯-¤+1)}\right]$$

$$-\frac{1}{¯-¤+1}\sum_{n=1}^{\infty}\frac{(-1)^n(¯-¤+n+1)\Gamma(n)}{(¯-¤)^n}$$

$$¯>¤. \qquad (4.5)$$

## 5. STOCHASTIC ORDERING

A random variable X is said to be smaller than a random variable Y in the

i. Stochastic order $(X \le_{st} Y)$ if $F_X(x)\ge F_Y(x)$ for all x

ii. Hazard rate order $(X \le_{hr} Y)$ if $h_X(x)\ge h_Y(x)$ for all x

iii. Mean residual life order $(X \le_{mrl} Y)$ if $m_X(x)\le m_Y(x)$ for all x

iv. Likelihood ratio order $(X \le_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

Shaked and Shanthikumar (1994) considered the following results for establishing stochastic ordering of distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y$$
$$\Downarrow$$
$$X \le_{st} Y \qquad (5.1)$$

**Theorem 5.1:** Let x follows the WLD $(¯_1,¤_1)$ and Y follows the WLD$(¯_2,¤_2)$. If $¤_1=¤_2$ & $¯_1>¯_2$ (or if $¯_1=¯_2$ & $¤_1<¤_2$), then

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \Rightarrow X \le_{st} Y$$

**Proof:** Consider the pdf of WLD in equation (2.1) and let

$$\frac{f_X(x)}{f_Y(x)}=\left(\frac{¯_1-¤_1}{¯_2-¤_2}\right)^2\left(\frac{¯_2-¤_2+1}{¯_1-¤_1+1}\right)e^{-x(¯_1-¯_2-¤_1+¤_2)}$$

$$\log\frac{f_X(x)}{f_Y(x)}=2\log\left(\frac{¯_1-¤_1}{¯_2-¤_2}\right)+\log\left(\frac{¯_2-¤_2+1}{¯_1-¤_1+1}\right)$$
$$-x(¯_1-¯_2-¤_1+¤_2)$$

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}=-(¯_1-¯_2-¤_1+¤_2) \qquad (5.2)$$

**Case (i)** if $¤_1=¤_2$ & $¯_1>¯_2$, then

$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}<0$. This means that

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \Rightarrow X \le_{st} Y$$

**Case (ii)** if $¯_1=¯_2$ & $¤_1<¤_2$, then

$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}<0$. This means that

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \Rightarrow X \le_{st} Y$$

Hence the weighted Lindley distribution is ordered strongly with respect to the stochastic ordering.

**Lemma:** Let $f(x)$ is twice differentiable density function of a continuous random variable X from the weighted Lindley distribution:

$$¦(x)=\frac{-f'(x)}{f(x)} \qquad (5.3)$$

i. If $¦'(x)<0$; for all $x>0$, then hazard rate is monotonically decreasing (DFR)

ii. If $¦'(x)>0$; for all $x>0$, then hazard rate is monotonically increasing (IFR)

$$¦(x)=\frac{-[1-(¯-¤)(1+x)]}{(1+x)} \qquad (5.4)$$

$$¦'(x)=\frac{1}{(1+x)^2}>0 \qquad (5.5)$$

Hence the hazard rate of the WLD is monotonically increasing (IFR).

**Theorem 5.2:** If $f(x)$ is probability density function of a continuous random variable X from the weighted Lindley distribution and the cdf of the WLD is

$F(x)=\int_0^x f(u)du$ then show that

$$\frac{d¤(x)}{f(x)}=\frac{x}{«'_1}, \text{ where } ¤(x)=\frac{1}{«'_1}\int_0^x uf(u)du . \qquad (5.6)$$

**Proof:** Let x be a continuous random variables with pdf $f(x)$ form WLD then

$$¤(x)=\frac{1}{«'_1}\int_0^x uf(u)du$$

After simplifications we get

$$¤(x)=1-e^{-x(¯-¤)}-x(¯-¤)e^{-x(¯-¤)}$$
$$-\frac{x^2(¯-¤)^2}{(¯-¤+2)}e^{-x(¯-¤)}.$$

and

$$d¤(x)=\frac{(¯-¤)^3}{(¯-¤+2)}x(1+x)e^{-x(¯-§)}$$

Finally after some simplification we get the expression in equation (5.6) as

$$\frac{d¤(x)}{f(x)}=x\frac{(¯-¤+1)(¯-¤)}{(¯-¤+2)}=\frac{x}{«'_1}$$

## 6. CONCLUSION

In this paper weighted Lindley distribution (WLD) is introduced by using a new weight function. The graphs shows that the WLD distribution is positively skewed moreover it is observed that mode is always less than from mean of the WLD. Some basic properties as moments, coefficient or skewness and kurtosis, median, quantile function and fisher information index of WLD have been derived. Some reliability measures such as survival function, cumulative hazard rate function, failure rate function and mean residual life of WLD are developed with graphs. The graphs show that the survival function and mean residual life of WLD are in decreasing trend. While the graphs of failure rate function are in monotonically increasing trend. Lemma has also been showed that the hazard rate function of WLD is monotonically increasing. Furthermore stochastic ordering of WLD is proved through a theorem.

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