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RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF RECORD VALUES FROM SIZE-BIASED PARETO DISTRIBUTION

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ABSTRACT

In this paper some recurrence relations satisfied by single and product moments of upper record values from size-biased Pareto distribution(S-BPD) are presented. Further a relation between moments of size-biased Pareto distribution and upper record values from size-biased Pareto distribution is established. Maximum likelihood estimators (MLE) of S-BPD are also presented. Confidence intervals and record quantile from size-biased Pareto distribution, based on upper record values have also been developed.

KEY WORDS: *Size-biased distribution; recurrence relations; moments; record values; confidence intervals; quantile; MLE; Pareto distribution; S-BPD.*

1.INTRODUCTION

Record values appear in many statistical applications. Record values involve in many real life applications including data relating to sports, climate, economics, students grade sheets, purchase order, memos and any other type of documents. Chandler (1952) formulated the theory of record values arising from a sequence of independently identically distributed continuous random variables and has now spread in various directions. Balakrishnan et al. (1995) obtained the recurrence relations for moments of record values for Gumbel distribution. Balakrishnan and Chan (1998) discussed the associated inference for the normal record values. Sultan et al. (2007) introduced the estimation and prediction from gamma distribution based on record values. Khan and Zia (2009) established recurrence relation of single and product moment from Gompertz distribution and they also presented a characterization. The recurrence relations for the single and double moments of record values can be used to calculate the different moments for any order and sample size in a simple reverting manner. The recurrence relations reduce the round-off error for calculating the moments compare with the numerical integration techniques. When we use the recurrence relations to calculate the moments, we need only few initial moments to be numerically calculated. Bashir and Ahmad (2014) developed record values from the inverse Gaussian distribution (IGD) including its various properties. They developed a limiting theorem and a recurrence relation of record values from the IGD. Bashir and Akhtar (2014) introduced the record values from the size-biased student's t distribution. They developed some properties of the record values from student's t distribution including some reliability measures.

Let $X_{U(1)} < X_{U(2)} < \ldots < X_{U(n)}$. be the upper record, Ahsanullah (1995) gave the following pdf of upper record values $X_{U(n)}$ is

$$f_n(x)=\frac{1}{(n-1)!}[-\ln(1-F(x))]^{n-1} f(x), \qquad \sim >0, \quad \sim < x<\infty. \tag{1.1}$$

The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ is

$$f_{i,j}(x,y)=\frac{1}{(i-1)!(j-i-1)!}[-\ln(1-F(x))]^{i-1}$$
$$\frac{f(x)}{1-F(x)}[-\ln(1-F(y))+\ln(1-F(x))]^{j-i-1}$$
$$f(y), \qquad \alpha < x < y < \infty. \tag{1.2}$$

Weighted distributions used when the recorded observations are not generated randomly. Moment distributions have a number of applications in forestry, in observational studies of human life, environment, insect, plant etc. Length-biased distributions have been used as moment distributions in reliability perspective. Size biased distributions are special case of the weighted distributions (Moment Distributions). Such distributions rise certainly in practice when observations from a sample are recorded with unequal probability. The weighted distributions arise when the observations generated from a stochastic process, are recorded according to some weighted function. When the weight function depends on the lengths of the items of concern then the subsequent distribution is called length biased.

Let the random variable X have distribution $f(x;\theta)$, with unknown parameter θ, and then the corresponding moment distribution is of the form

$$g(x;\bar{\ }) = \frac{w(X)f(x;\bar{\ })}{E(w(X))}, \tag{1.3}$$

where $w(X)$ is a non-negative weight function such that $E[w(X)]$ exists.

When the weight function has the form, $w(X) = x^m$ then such distributions are named as size-biased distributions of order m and are written as [Patil and Ord, (1976); Patil(1981); Mahfoud and Patil, (1982)]:

$$g(x;\bar{\ }) = \frac{x^m f(x;\bar{\ })}{«'_m}, \tag{1.4}$$

where μ'_m is the m^{th} raw moment of $f(x;\theta)$.

When $m = 1 \, or \, 2$, these special cases are termed as length-biased or size-biased distribution and area-biased distribution, respectively. Patil and Rao (1978) observed some general models leading to weighteddistributions with weight functions not essentially limited by unity. The results were useful to the analysis of data relating to human populations and wildlife management. Patil (2002) provided detailed discussion on weighted distribution in the context of size-biased and length-biased distributions and their applications. Mir and Ahmad (2009) presented some size-biased probability distributions and their generalizations. These set of distributions offer a linking approach for the problems where the observations fall in the non-experimental, non- replicated, and nonrandom categories. They presented some of the possible applications of size- biased distribution theory to some real life data. The sized-biased Pareto distribution was introduced by Dara and Ahmad (2011). The probability density function of size-biased Pareto distribution is

$$f(x)=(\ \ -1)\sim^{\ -1}x^{-} \qquad \sim >0, \quad >0, \sim < x<\infty.$$

2. RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS

The cdf of size-biased Pareto distribution (S-BPD) is

$$F(x) = 1 - \sim^{\ -1} / x^{\ -1}. \tag{2.1}$$

The relation between pdf and cdf of size-biased Pareto distribution (S-BPD) is

$$f(x) = \frac{(\ \ -1)[1-F(x)]}{x}. \tag{2.2}$$

The relation (2.2) will be used to derive some simple recurrence relations for single and product moments of upper record values from S-BPD.

Let $X_{U(1)} < X_{U(2)} < \ldots < X_{U(n)}$. be the upper record values from the size-biased Pareto distribution. Then the pdf of upper record value from S-BPD is

$$f_n(x)=\frac{1}{(n-1)!}[-\ln(1-F(x))]^{n-1} f(x), \quad \sim >0, \sim < x<\infty. \tag{2.3}$$

The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ from the S-BPD

$$f_{i,j}(x,y)=\frac{1}{(i-1)!(j-i-1)!}\left[-\ln(1-F(x))\right]^{i-1}$$
$$\frac{f(x)}{1-F(x)}\left[-\ln(1-F(y))+\ln(1-F(x))\right]^{j-i-1}$$
$$f(y), \qquad \sim < x < y < \infty. \tag{2.4}$$

Theorem 1: For $n \geq 1$ and $r = 1,2,3,.....$

$$E\left(X^r_{U(n)}\right)=\left(1-\frac{r}{\ -1}\right)^{-1} E\left(X^r_{U(n-1)}\right) \tag{2.5}$$

Proof: From (2.4), consider for and $r = 1,2,3,.....$

$$E\left(X^r_{U(n)}\right)=\frac{1}{(n-1)!}\int_{\alpha}^{\infty} x^r\left[-\ln(1-F(x))\right]^{n-1} f(x)dx$$

$$=\frac{1}{(n-1)!}\int_{\alpha}^{\infty} x^r\left[-\ln(1-F(x))\right]^{n-1} \frac{(\beta-1)(1-F(x))}{x}dx$$

$$=\frac{(\beta-1)}{r}\int_{\alpha}^{\infty}\frac{1}{(n-1)!} x^r\left[-\ln(1-F(x))\right]^{n-1} f(x)dx$$
$$-\frac{(\beta-1)}{r}\int_{\alpha}^{\infty}\frac{1}{(n-2)!} x^r\left[-\ln(1-F(x))\right]^{n-2} f(x)dx$$

$$=\frac{(\beta-1)}{r}\left[E\left(X^r_{U(n)}\right)-E\left(X^r_{U(n-1)}\right)\right]$$

where the last but one step follows by the integration by parts. The recurrence relation in equation (2.5) is obtained by rewriting the above equation.

Theorem 2: For $j \geq i+2$ and $r,s = 1,2,3,......$

$$E\left(X^r_{U(i)}X^s_{U(j)}\right)=\left(1-\frac{s}{\beta-1}\right)^{-1} E\left(X^r_{U(i)}X^s_{U(j-1)}\right) \tag{2.6}$$

and for $j = i+1$

$$E\left(X^r_{U(i)}X^s_{U(i+1)}\right)=\left(1-\frac{s}{\beta-1}\right)^{-1}\left[E\left(X^{r+s}_{U(i)}\right)\right]. \tag{2.7}$$

Proof: From equation (2.5)

$$E\left(X^r_{u(i)}X^s_{U(j)}\right)=\iint_{\alpha<x<y<\infty} x^r y^s f_{i,j}(x,y)dxdy$$

$$=\frac{1}{(i-1)!(j-i-1)!}\int_{\alpha}^{\infty} x^r\left[-\ln(1-F(x))\right]^{i-1}\frac{f(x)}{1-F(x)}\mathrm{I}(x)dx, \tag{2.8}$$

where

$$\mathrm{I}(x)=\int_{x}^{\infty} y^s\left[-\ln(1-F(x))+\ln(1-F(x))\right]^{j-i-1} f(y)dy$$

Using the relation in equation (2.2), we have

$$\mathrm{I}(x)=\int_{x}^{\infty} y^s\left[-\ln(1-F(x))+\ln(1-F(x))\right]^{j-i-1}$$
$$\frac{(\ -1)(1-F(y))}{y}dy$$

,

Integrating by parts, we get

$$\mathrm{I}(x)=\frac{(\ -1)}{s}\int_{x}^{\infty} y^s\left[-\ln(1-F(x))+\ln(1-F(x))\right]^{j-i-1} f(y)dy$$
$$-\frac{(j-i-1)(\ -1)}{s}$$
$$\int_{x}^{\infty} y^s\left[-\ln(1-F(x))+\ln(1-F(x))\right]^{j-i-2} f(y)dy.$$

Substituting the expression of $\mathrm{I}(x)$ in equation (2.8) and simplifying the resulting equations, we obtain

$$E\left(X^r_{U(i)}X^s_{U(j)}\right)=\frac{(\ -1)}{s}\left[\begin{matrix}E\left(X^r_{U(i)}X^s_{U(j)}\right)- \\ E\left(X^r_{U(i)}X^s_{U(j-1)}\right)\end{matrix}\right]. \tag{2.9}$$

The recurrence relations in (2.6) and (2.7) are derived by rewriting the equation (2.9).

2.1.1 Relation between rth Moments:-

If $X_{U(1)} < X_{U(2)} < \ldots < X_{U(n)}$. be the upper record values from the S-BPD with pdf

$$f_n(x)=\frac{(\ -1)\sim^{-1}}{(n-1)!}x^{-}\left(-\ln\left(\sim^{-1}x^{1-}\right)\right)^{n-1},$$
$$\sim > 0, \quad \sim < x < \infty, \tag{2.10}$$

and the size-biased Pareto distribution with pdf in (1.3), then a relation between rth moments of size-biased Pareto distribution and upper record values from S-BPD pdf is exists as

$$\ll'_{r(n)} = \sim^{-r(n-1)}\left(\ll'_r\right)^n. \tag{2.11}$$

Where, $«'_r$ are the rth moments of size-biased Pareto distribution and $«'_{r(n)}$ are the rth moments of upper record from S-BPD.

$$«'_r = \frac{(1- \quad)\sim^r}{(1- \quad + r)}.$$

$$«'_{r(n)} = \frac{(1- \quad)^n \sim^r}{(1- \quad + r)^n}.$$

From this relationship we can derive moments of upper record values from the S-BPD using the rth moments of parent size-biased Pareto distribution.

3. MAXIMUM LIKELIHOOD ESTIMATION

According to Ahsanullah and Houchens (1989), let $X_{U(1)} < X_{U(2)} < \ldots < X_{U(n)}$. be the upper record values from the size-biased Pareto distribution, let us denote $X_{U(i)}$ by X_i, $i = 1,2,\ldots,n.$. Considering the pdf of nth upper record value in (2.3), where $f(.)$ is given by (1.5), and $F(.)$ is the corresponding cdf. Then likelihood function in this case is

$$L(\sim, \quad) = (\quad -1)^n \sim^{n(\quad -1)} x_n^{1-} \quad / \prod_{i=i}^{n} x_i ,$$

$$\sim < x_{U(0)} < x_{U(1)} < \ldots.. < x_{U(n)},$$

Therefore log-likelihood function is

$$l(\sim, \quad) = n\ln(\quad -1) + n(\quad -1)\ln\sim + (1- \quad)\ln x_n - \sum_{i=1}^{n} \ln x_i \qquad (3.1)$$

$$\sim < x_{U(0)} < x_{U(1)} < \ldots.. < x_{U(n)},$$

so the MLE for $\sim$ is

$$\hat{\sim} = x_{U(0)} = x_1$$
$$\hat{\sim} = x_1 \qquad (3.2)$$

from (3.1), MLE for is

$$\hat{\ } = 1 + \frac{n}{\ln\left(X_{U(n)} / X_{U(0)}\right)}$$

or

$$\hat{\ } = 1 + \frac{n}{\ln\left(x_n / x_1\right)} \qquad (3.3)$$

Clearly, $\hat{\sim}$ is distributed as size-biased Pareto distribution

Therefore,

$$E(\hat{\sim}) = \frac{(\quad -1)\sim}{(\quad -2)} \qquad (3.4)$$

and

$$\text{MSE}(\hat{\sim}) = \frac{2(2 \quad -3)\sim^2}{(\quad -2)^2(\quad -3)} \qquad (3.5)$$

4. CONFIDENCE INTERVALS

Let $X_{U(n)}$ and $X_{L(n)}$ be, respectively, the nth upper and lower record statistics from a family with cdf $F(.)$. Then $100(1-g)$% confidence interval for the upper record values and lower record values are given by Teimouri and Gupta (2012).

$$\left[F^{-1}\left(1-\exp\left(-G_n^{-1}(g/2)\right)\right), F^{-1}\left(1-\exp\left(-G_n^{-1}(1-g/2)\right)\right)\right]. \qquad (4.1)$$

$$\left[F^{-1}\left(\exp\left(-G_n^{-1}(1-g/2)\right)\right), F^{-1}\left(\exp\left(-G_n^{-1}(g/2)\right)\right)\right]. \qquad (4.2)$$

$$G_n^{-1}(g) = \min\{x / G_n(x) > g\},$$

G_n , is continuous random variable having gamma distribution with shape parameter $n > 0$ and its pdf is

$$f_{G_n}(x) = \frac{x^{n-1}\exp(-x)}{\Gamma n}, \qquad x > 0.$$

The $100(1-g)$% confidence intervals for the given upper record values from the S-BPD by using (4.1)

$$F_{X_{U(n)}}(y) = P(G_n < -\ln(1 - F(y))),$$

$$= P\left(G_n < -\ln\left(\sim^{\quad -1} y^{1-} \quad\right)\right).$$

$$\left\{\sim\left(\exp\left(-G_n^{-1}(g/2)\right)\right)^{\frac{1}{1-}}, \sim\left(\exp\left(-G_n^{-1}(1-g/2)\right)\right)^{\frac{1}{1-}}\right\}. \qquad (4.3)$$

Now by using (4.2) we obtain the $100(1-g)$% confidence intervals for the lower record values from size-biased Pareto distribution,

$$F_{X_{L(n)}}(y) = P(G_n > -\ln F(y)),$$
$$= P\left(G_n > -\ln\left(1 - \sim^{\quad -1} y^{1-} \quad\right)\right).$$

$$\left\{\sim\left(1-\exp\left(-G_n^{-1}(g/2)\right)\right)^{\frac{1}{1-}}, \sim\left(1-\exp\left(-G_n^{-1}(1-g/2)\right)\right)^{\frac{1}{1-}}\right\}. \qquad (4.4)$$

where, $0 < g < 1$.

4.1. Record Quantile:-

Teimouri and Gupta (2012) derived pth quantile for $0 < p < 1$ of nth upper and lower record statistics respectively, is given by

$$q_{U(n)}(p) = F^{-1}\left(1 - \exp\left(-G_n^{-1}(p)\right)\right). \quad (4.5)$$

$$q_{L(n)}(p) = F^{-1}\left(\exp\left(-G_n^{1}(1-p)\right)\right). \quad (4.6)$$

Now by using (4.5) and (4.6) , we obtain the upper and lower record quantile for the size-biased Pareto distribution, respectively

$$q_{U(n)}(p) = \sim \left(\exp\left(-G_n^{-1}(p)\right)\right)^{\frac{1}{1-}} \quad . (4.7)$$

$$q_{L(n)}(p) = \sim \left(1 - \exp\left(-G_n^{-1}(1-p)\right)\right)^{\frac{1}{1-}} \quad . (4.8)$$

where $0 < p < 1$.

5. CONCLUSION

In this paper some recurrence relations for single and product moments of record values from the size-biased Pareto distribution have been established. A relation between moments of size-biased Pareto distribution and size-biased Pareto upper record value distribution has been derived. By using this relation, moments of upper record values from size-biased Pareto distribution can be determined. Confidence intervals and record quantile for the upper record values from size-biased Pareto distribution have been introduced. Similar results for Weibull distribution have been obtained by Teimouri and Gupta (2012). Maximum likelihood estimators of the parameters of size-biased Pareto distribution from upper record values are developed. Further mean and MSE of MLE parameter of size-biased Pareto distribution are also derived.

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