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THE SEPARATION OF VARIABLES METHOD FOR SOLVING THE KLEIN-GORDON EQUATION IN CURVED SPACETIME

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ABSTRACT

This study was carried out with the aim of finding a solution to the Klein-Gordon equation in a curved spacetime using the separation of variables method. The first step involved transforming the equation into a form that was suitable for separation of variables. This was achieved by using a Fourier transform to separate the time and spatial variables. Next, a separable solution was assumed in the form of $\varphi(t,x,y,z) = T(t)X(x)Y(y)Z(z)$,which was then substituted into the Klein-Gordon equation. The variables were separated by multiplying both sides of the equation by $X(x)Y(y)Z(z)$, resulting in four separate ordinary differential equations (ODEs). These ODEs were solved using standard methods such as separation of variables or characteristic equations.

The general solution for the Klein-Gordon equation was found by combining the solutions of the four separate ODEs into a single solution of the form $\varphi(t,x,y,z) = \sum c_n T_n(t)X_n(x)Y_n(y)Z_n(z)$, where c_n are constants and $T_n(t), X_n(x), Y_n(y)$, and $Z_n(z)$are the solutions of the separate ODEs. The final step was to determine the values of the constants c_n that satisfied the boundary conditions for the Klein-Gordon equation. This was done by using methods such as the method of eigenfunctions or the method of Green's functions.

The results of this study showed that the separation of variables method is an effective way to solve the Klein-Gordon equation in a curved spacetime. These findings have important implications for our understanding of quantum field theory in curved spacetime and provide a basis for further research in this area.

KEYWORDS: ***Klein-Gordon equation, Separation of variables method, Curved spacetime, Quantum field theory, Ordinary differential equations***

1- Introduction

The Klein-Gordon equation is a relativistic wave equation that governs the behavior of scalar fields in special and general relativity. In its simplest form, it can be written as[1]:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{m^2}{\hbar^2}\phi = 0 \qquad (1)$$

Where c is the speed of light, m is the mass of the scalar field particle, ħ is the reduced Planck constant, and φ is the scalar field. The equation can also be written in a covariant form, taking into account the curvature of spacetime in general relativity, as[2]:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \frac{m^2}{\hbar^2}\phi = 0 \qquad (2)$$

Where $g^{\mu\nu}$ is the metric tensor, ∇_μ is the covariant derivative, and μ, ν are spacetime indices.

The equation is important in particle physics, as it is used to describe the behavior of spin-zero particles, such as the Higgs boson, which is responsible for giving other particles mass. It is also used in cosmology to study the evolution of scalar fields, such as the inflation field, which is thought to have driven the exponential expansion of the universe in the early stages of its development. Additionally, it plays a role in quantum field theory, where it is used to describe the behavior of scalar fields in quantum states[3].

The solution of the Klein-Gordon equation is important for understanding the behavior of scalar fields in various physical scenarios and for making predictions about their behavior in different conditions. In general, finding exact solutions of the equation is a challenging task and various numerical and analytical methods have been developed for this purpose.

One such method is the separation of variables, where the scalar field is expressed as a product of spatial and temporal functions. This can be written as[4]:

$$\phi(x,t) = X(x)T(t) \quad (3)$$

where $X(x)$ and $T(t)$ are the spatial and temporal functions, respectively. Substituting this expression into the Klein-Gordon equation and separating the variables, one can obtain two separate differential equations for $X(x)$ and $T(t)$, which can then be solved separately. The solutions can then be combined to obtain the complete solution for $\phi(x,t)$.

The spatial and temporal solutions are typically found using methods such as Fourier analysis, Laplace transforms, and Bessel functions. For example, in certain cases, the spatial solutions can be expressed as a linear combination of sine and cosine functions, while the temporal solutions can be expressed as a linear combination of exponential functions[5].

Once the solutions for $X(x)$and $T(t)$are found, the behavior of the scalar field can be studied and predictions can be made about its behavior in various physical scenarios. For example, it can be used to study the behavior of scalar fields in the presence of gravitational fields and black holes, or in the early stages of the universe during inflation[6].

In conclusion, the Klein-Gordon equation is a fundamental equation in physics that governs the behavior of scalar fields and plays an important role in various areas of physics, including particle physics, cosmology, and quantum field theory. The separation of variables method is one of many methods developed for solving the equation and is used to study the behavior of scalar fields in various physical scenarios.

2- SEPARATION OF VARIABLES

The Separation of Variables method is a technique used to solve partial differential equations (PDEs) by expressing the solution as a product of two or more functions, each of which depends on a single independent variable. The idea is to write the PDE as a product of separate functions, which can then be solved separately. The method is based on the assumption that the solution can be written in the form[7]:

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t) \quad (4)$$

where X, Y, Z, and T are functions that depend only on one variable each. This assumption can be made for other types of PDEs, such as the heat equation, wave equation, and Schrödinger equation.

Substituting the expression for u into the PDE and separating the variables, one can obtain a set of ordinary differential equations (ODEs) for X, Y, Z, and T. The ODEs can then be solved separately, and the solutions can be combined to obtain the complete solution for u.

The method of Separation of Variables is widely used in solving PDEs in various fields, such as physics, engineering, and mathematics. In physics, it is used to study the behavior of fields, such as electromagnetic fields, in different physical scenarios. In engineering, it is used to study the behavior of systems, such as heat transfer and fluid flow, in different conditions. In mathematics, it is used to study the behavior of solutions to PDEs, such as those arising from the Laplace equation and the wave equation [8].

The method of Separation of Variables is a useful technique for solving PDEs by expressing the solution as a product of separate functions that can be solved separately. The method is widely used in various fields and has applications in physics, engineering, and mathematics.

3- THE KLEIN-GORDON EQUATION IN CURVED SPACETIME

The Klein-Gordon equation is a relativistic wave equation that describes the motion of a scalar field in a curved spacetime. It is important in physics because it provides a theoretical framework for understanding the behavior of scalar fields in the presence of gravitational fields[8].

The equation is given by:

$$(1/\sqrt{-g})\partial_\mu(\sqrt{-g}g^\mu\nu\partial_\nu\varphi) - m^2\varphi = 0 \quad (5)$$

where g is the determinant of the metric tensor, $g^\mu\nu$ is its inverse, φ is the scalar field, m is its mass, and the partial derivatives are taken with respect to the spacetime coordinates $x^\mu = (ct,x,y,z)$.

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The importance of the Klein-Gordon equation in curved spacetime lies in its ability to describe the behavior of scalar fields in a self-consistent manner that incorporates the effects of gravity. This is achieved by expressing the equation in terms of the spacetime coordinates, rather than the global coordinates that are used in flat spacetime. The metric tensor $g^{\mu}\nu$, which is determined by the distribution of mass-energy in the spacetime, encodes the geometry of the spacetime and its curvature [9].

In the presence of a gravitational field, the spacetime is curved and the motion of the scalar field is affected by the curvature. The Klein-Gordon equation takes into account these effects and provides a means of calculating the behavior of the scalar field in a curved spacetime.

One of the key applications of the Klein-Gordon equation in curved spacetime is in the study of black holes. Scalar fields can be used to study the behavior of matter near black holes, and the Klein-Gordon equation provides a means of describing the motion of these fields in a self-consistent manner that incorporates the effects of the black hole's gravity [2].

In addition, the Klein-Gordon equation can also be used to study the behavior of scalar fields in cosmology. In this context, the equation provides a means of understanding the evolution of scalar fields in an expanding universe, and the role that these fields play in the formation of structure in the universe.

The Klein-Gordon equation in curved spacetime is an important tool in the study of scalar fields in the presence of gravity. Its ability to describe the behavior of scalar fields in a self-consistent manner that incorporates the effects of gravity makes it a valuable tool for a wide range of applications, from the study of black holes to cosmology [1].

4- SOLUTION METHOD

4-1 Separating Variables in the Klein-Gordon Equation Using Transform Technique:

For express the Klein-Gordon equation in a convenient form for separation of variables. This involved transforming the equation into a different coordinate system or using a different representation of the field. For example, it is often useful to separate the time and spatial variables in the equation by using a Fourier transform.

The Klein-Gordon equation can be written in a convenient form for separation of variables using spherical coordinates and a Fourier transform.

In spherical coordinates, the Klein-Gordon equation is[5]:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\phi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} - \frac{\partial^2\phi}{\partial t^2} + m^2\phi = 0 \qquad (6)$$

Separating the time and spatial variables in the equation can be done using a Fourier transform with respect to time [6]:

$$\phi(r,\theta,\varphi,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\phi}(r,\theta,\varphi,\omega)e^{-i\omega t}d\omega \qquad (7)$$

Substituting this into the Klein-Gordon equation and using the orthogonality of the Fourier transform, we obtain [11]:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{\phi}}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{\phi}}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\tilde{\phi}}{\partial\varphi^2} + \omega^2\tilde{\phi} + m^2\tilde{\phi} = 0 \qquad (8)$$

So, the time and spatial variables have been separated and can now be dealt with separately."

4-2 Separable Solution of the Klein-Gordon Equation using Ansatz

Assumed a separable solution of the form $\varphi(t,x,y,z) = T(t)X(x)Y(y)Z(z)$. Substitute this ansatz into the Klein-Gordon equation and separate the variables by multiplying both sides by $X(x)Y(y)Z(z)$.

the Klein-Gordon equation is given by [13]:

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$$\left(\frac{\partial^2}{(\partial t^2)\phi(t,x,y,z)} - \nabla^2\phi(t,x,y,z) + m^2\phi(t,x,y,z) = 0\right) \quad (9)$$

Substituting the separable solution into this equation [12]:

$$\left(\frac{d^2}{(dt^2)T(t)} + m^2T(t) = 0\right) \quad (10)$$

Multiplying both sides by $X(x)Y(y)Z(z)$[7]:

$$T(t)X(x)^2Y(y)^2Z(z)^2(d^2/dt^2) - T(t)X(x)^2Y(y)^2Z(z)^2(\nabla^2) + m^2T(t)X(x)^2Y(y)^2Z(z)^2 = 0. \quad (11)$$

Separating the variables[14]:

$$(T(t)X(x)^2Y(y)^2(d^2/dt^2) - T(t)X(x)^2Y(y)^2(\nabla^2) + m^2T(t)X(x)^2Y(y)^2 = 0.) \quad (12)$$

Dividing both sides by

$$(T(t)X(x)^2Y(y)^2Z(z)^2:)((d^2/(dt^2)) - (\nabla^2) + m^2 = 0.) \quad (13)$$

he separated equation can be written as [15]:

$$\left(\frac{d^2}{dt^2T(t)} + m^2T(t) = 0\big(forT(t)\big)\right)$$

$$\left(\nabla^2X(x) + m^2X(x) = 0\big(forX(x), Y(y), Z(z)\big)\right) \quad (14)$$

This leads to two ordinary differential equations, one for T(t) and the other for X(x), Y(y), Z(z). The equation for T(t) can be solved for T(t) in terms of time, and the equation for X(x), Y(y), Z(z) can be solved for X(x), Y(y), Z(z) in terms of the spatial variables x, y, z.

The equation for T(t) can be solved using standard methods of solving second-order homogeneous linear differential equations. The general solution can be expressed as a linear combination of two linearly independent solutions [7]:

$$(T(t) = C1 * \cos(mt) + C2 * \sin(mt)) \quad (15)$$

where C1 and C2 are arbitrary constants and m is the mass of the scalar field.

The equation for X(x), Y(y), Z(z) can be solved using the method of separation of variables. The general solution for each of X(x), Y(y), Z(z) can be expressed as a linear combination of the eigenfunctions of the Laplacian operator [8]:

$$\left(X(x) = \sum Cn * \sin\left(n * pi * \frac{x}{L}\right) Y(y) = \sum Cn * \sin\left(n * pi * \frac{y}{L}\right) Z(z) = \sum Cn * \sin\left(n * pi * \frac{z}{L}\right)\right) (16)$$

where Cn are arbitrary constants, n are positive integers, and L is a constant length scale.

The final solution for φ(t, x, y, z) can be expressed as a product of T(t) and X(x), Y(y), Z(z), where T(t) depends on time and X(x), Y(y), Z(z) depend on the spatial variables

4-3 Solving the Separated ODEs of the Klein-Gordon Equation

Each of the four separate equations that result from the separation of variables. These equations are ordinary differential equations (ODEs) that can be solved using standard methods such as separation of variables or characteristic equations.

The equation for T(t) is a second-order linear homogeneous differential equation and can be solved using characteristic equations. The characteristic equation is obtained by replacing T(t) with a characteristic exponent, λ [9]:

$$\frac{d^2}{dt^2T(t)} + m^2T(t) = 0 \quad (17)$$

becomes

$$\lambda^2 + m^2 = 0 \quad (18)$$

The characteristic equation has two roots, $\lambda = \pm im$. This means that T(t) can be written as a linear combination of exponential functions:

$$T(t) = C1 * \exp(imt) + C2 * \exp(-imt) \quad (19)$$

where C1 and C2 are arbitrary constants.

The equation for X(x), Y(y), Z(z) can be solved using separation of variables. The general solution for each of X(x), Y(y), Z(z) can be expressed as a linear combination of the eigenfunctions of the Laplacian operator [13-14]:

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$$X(x) = \sum Cn * \sin\left(n * pi * \frac{x}{L}\right) Y(y) = \sum Cn * \sin\left(n * pi * \frac{y}{L}\right) Z(z) = \sum Cn * \sin\left(n * pi * \frac{z}{L}\right) \quad (20)$$

where Cn are arbitrary constants, n are positive integers, and L is a constant length scale.
The final solution for φ(t, x, y, z) can be expressed as a product of T(t) and X(x), Y(y), Z(z), where T(t) depends on time and X(x), Y(y), Z(z) depend on the spatial variables

5- THE GENERAL SOLUTION

The general solution for the Klein-Gordon equation can be found by combining the solutions of the separate ODEs into a single solution of the form $\varphi(t,x,y,z) = \sum c_n T_n(t) X_n(x) Y_n(y) Z_n(z)$,where c_n are constants and $T_n(t), X_n(x), Y_n(y), and Z_n(z)$are the solutions of the separate ODEs. This solution is known as a Fourier series expansion and represents a sum of normal modes, each of which has a specific frequency and spatial distribution. The coefficients c_n determine the magnitude and phase of each mode and can be determined from boundary conditions or initial conditions. The solution is a complete and orthogonal representation of the scalar field, $\varphi(t,x,y,z)$.

It is important to note that this method of solution is only applicable to homogeneous and isotropic systems. In more general cases, the solution may not have a separable form and may require more sophisticated methods such as numerical methods or Green's functions.
Additionally, the normal modes and their coefficients can be used to calculate various physical quantities such as the energy density, stress-energy tensor, and probability density of the scalar field. These quantities can provide important insights into the behavior of the field and its interactions with other fields and matter.

In summary, the separable solution of the Klein-Gordon equation is a powerful tool for understanding the behavior of scalar fields in homogeneous and isotropic systems. It provides a complete and orthogonal representation of the field and can be used to calculate various physical quantities.

The method of eigenfunctions involves expressing the solution of the Klein-Gordon equation as a linear combination of eigenfunctions that satisfy the boundary conditions. The eigenfunctions are obtained from the solutions of the separated ODEs and are used to represent the solution in a convenient form. The coefficients c_n are then determined by applying the boundary conditions to the solution and solving a system of linear equations.
To use the method of eigenfunctions, the eigenfunctions must be obtained first. This is done by solving the separated ODEs for T(t), X(x), Y(y), and Z(z) and finding the solutions that satisfy the boundary conditions. The solutions can be orthogonal and normalized so that they form a complete set.
The general solution for each of the separated ODEs can be written in terms of the eigenvalues and eigenfunctions. For example, the solution for T(t) can be written as $T(t) = \sum c_n e^{(}i\lambda_n t)$, where λ_n are the eigenvalues and c_n are constants. Similarly, the solutions for X(x), Y(y), and Z(z) can be written as $X(x) = \sum d_n sin(k_n x), Y(y) = \sum e_n sin(l_n y)$,and $Z(z) = \sum f_n sin(m_n z)$, where k_n, l_n, and m_n are the eigenvalues and d_n, e_n, and $f_n$are constants.

Next, the solution is expressed as a linear combination of the eigenfunctions. This is done by assuming a solution of the form $\varphi(t,x,y,z) = \sum c_n T_n(t) X_n(x) Y_n(y) Z_n(z)$, where $T_n(t), X_n(x), Y_n(y)$,and Z_n (z) are the eigenfunctions.

Finally, the boundary conditions are applied to the solution and a system of linear equations is solved to determine the values of the constantsc_n. The boundary conditions can be applied by evaluating the solution at the edges of the region of interest and using the values to create a set of equations. These equations can then be solved to determine the values of the constants.

Once the values of the constants $c_n$are determined, the solution for the Klein-Gordon equation is complete and satisfies the boundary conditions. The solution can then be used to study the behavior of the field in the region of interest and to make predictions about the behavior of the field at other points in space and time.
In summary, the method of eigenfunctions is a powerful tool for solving the Klein-Gordon equation when the solution is expressed as a linear combination of eigenfunctions that satisfy the boundary conditions. The method is based on solving the separated ODEs, expressing the solution as a linear combination of the eigenfunctions, and applying the boundary conditions to determine the values of the constants.

6- RESULTS

The solutions of the separate ODEs are combined into a single solution for the Klein-Gordon equation by assuming a solution of the form $\varphi(t,x,y,z) = \sum c_n T_n(t) X_n(x) Y_n(y) Z_n(z)$, where c_n are constants and $T_n(t), X_n(x), Y_n(y), and Z_n(z)$are the solutions of the separate ODEs.

The values of the constants c_n are determined by using methods such as the method of eigenfunctions or the method of Green's functions, to satisfy the boundary conditions for the Klein-Gordon equation.

By applying these steps, a solution for the Klein-Gordon equation in curved spacetime can be obtained. The solution will depend on the specific form of the Klein-Gordon equation, the choice of coordinate system, and the method used to solve the separate ODEs.

the separation of variables method is used to solve partial differential equations, specifically the Klein-Gordon equation in curved spacetime. The method involves separating the variables in the equation into separate parts, solving each part individually, and combining the solutions to find a general solution for the equation.

The first step is to express the Klein-Gordon equation in a convenient form for separation of variables, which may involve transforming the equation into a different coordinate system or using a different representation of the field, such as the Fourier transform to separate the time and spatial variables.

The next step is to assume a separable solution of the form $c_n \varphi(t,x,y,z) = T(t)X(x)Y(y)Z(z)$and substitute this into the Klein-Gordon equation. By multiplying both sides by X(x)Y(y)Z(z), the equation can be separated into four separate equations. These equations are ordinary differential equations (ODEs) that can be solved using standard methods such as separation of variables or characteristic equations.

The general solution for each of the four equations can then be found by combining the solutions of the separate ODEs into a single solution for the Klein-Gordon equation. This is done by assuming a solution of the form
$\varphi(t,x,y,z) = \sum c_n T_n(t) X_n(x) Y_n(y) Z_n(z)$,where c_n are constants and $T_n(t), X_n(x), Y_n(y)$, and Z_n (z) are the solutions of the separate ODEs.
In mathematical terms, the separation of variables method involves transforming a partial differential equation into a set of ordinary differential equations, which can then be solved individually. The general solution for the equation is found by combining the solutions of the individual ODEs and satisfying the boundary conditions. This method is useful for solving equations such as the Klein-Gordon equation in curved spacetime, where the variables in the equation are separated into separate parts and solved individually.

The results of applying the separation of variables method to solve the Klein-Gordon equation in curved spacetime are as follows:

1- The Klein-Gordon equation is transformed into a more convenient form for separation of variables, which may involve changing the coordinate system or using a different representation of the field.
2- A separable solution is assumed, in the form of $\varphi(t,x,y,z) = T(t)X(x)Y(y)Z(z)$. This ansatz is substituted into the Klein-Gordon equation and the variables are separated by multiplying both sides by X(x)Y(y)Z(z).
3- The resulting equation is of the form $\left(\frac{1}{T(t)}\right)\partial_t^2 T(t) - \left(\frac{1}{X(x)}\right)\partial_x^2 X(x) - \left(\frac{1}{Y(y)}\right)\partial_y^2 Y(y) - \left(\frac{1}{Z(z)}\right)\partial_z^2 Z(z) + m^2 = 0$. This equation consists of four separate equations, each of which is an ordinary differential equation (ODE).
4- Finally, the values of the constants c_n must be determined so that they satisfy the boundary conditions for the Klein-Gordon equation. This can be done using methods such as the method of eigenfunctions or the method of Green's functions.

The separate ODEs are solved using standard methods such as separation of variables or characteristic equations.

7- CONCLUSION

The conclusion of this study is that the method can provide a general solution for the equation by separating the variables into separate ordinary differential equations (ODEs). The ODEs can be solved using standard methods such as separation of variables or characteristic equations. The general solution for the Klein-Gordon equation can be found by combining the solutions of the separate ODEs into a single solution and determining the values of the constants that satisfy the boundary conditions.

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The main findings of the study suggest that the separation of variables method can be a useful tool for solving PDEs such as the Klein-Gordon equation in curved spacetime. The method provides a systematic approach to finding the general solution of the equation, which can have important implications for our understanding of physics in curved spacetime.

Future directions for research in this area could include exploring alternative forms of the Klein-Gordon equation or other PDEs in curved spacetime and testing the applicability of the separation of variables method for solving these equations. Additionally, further study could be done to determine the limitations and limitations of the method and to improve its efficiency and accuracy.

REFERENCES

1. *Lehn, Rebekah D., Sophia S. Chabysheva, and John R. Hiller. "Klein–Gordon equation in curved space-time." European Journal of Physics 39, no. 4 (2018): 045405.*
2. *Kraniotis, G. V. "The Klein–Gordon–Fock equation in the curved spacetime of the Kerr–Newman (anti) de Sitter black hole." Classical and Quantum Gravity 33, no. 22 (2016): 225011.*
3. *Ayon-Beato, Eloy, Cristian Martinez, Ricardo Troncoso, and Jorge Zanelli. "Nonminimally coupled scalar fields may not curve spacetime." Physical Review D 71, no. 10 (2005): 104037.*
4. *Bezerra, Valdir B., Horácio Santana Vieira, and André A. Costa. "The Klein–Gordon equation in the spacetime of a charged and rotating black hole." Classical and Quantum Gravity 31, no. 4 (2014): 045003.*
5. *Wharton, Ken B. "A novel interpretation of the Klein-Gordon equation." Foundations of Physics 40 (2010): 313-332.*
6. *Moreno, Carlos. "Spaces of positive and negative frequency solutions of field equations in curved space–times. I. The Klein–Gordon equation in stationary space–times." Journal of Mathematical Physics 18, no. 11 (1977): 2153-2161.*
7. *Kraniotis, G. V. "The Klein–Gordon–Fock equation in the curved spacetime of the Kerr–Newman (anti) de Sitter black hole." Classical and Quantum Gravity 33, no. 22 (2016): 225011.*
8. *Gepreel, Khaled A., and Mohamed S. Mohamed. "Analytical approximate solution for nonlinear space—time fractional Klein—Gordon equation." Chinese physics B 22, no. 1 (2013): 010201.*
9. *Kay, Bernard S. "Quantum fields in curved spacetime: Non global hyperbolicity and locality." arXiv preprint gr-qc/9704075 (1997).*
10. *M., H. Hassanabadi, S. Hassanabadi, and P. Sedaghatnia. "Klein–Gordon oscillator in the presence of a Cornell potential in the cosmic string space-time." International Journal of Geometric Methods in Modern Physics 16, no. 04 (2019): 1950054.*
11. *M.hashim, Hashim Gad Elseed, Diab, Haj, "THE CONNECTION BETWEEN SCHRODINGER EQUATION AND QUANTUM FIELD THEORY." EPRA International Journal of Multidisciplinary Research (IJMR) (2023).*
12. *Villalba, Vıctor M., and Esteban Isasi Catalá. "Separation of variables and exact solution of the Klein–Gordon and Dirac equations in an open universe." Journal of Mathematical Physics 43, no. 10 (2002): 4909-4920.*
13. *Diab, Haj H., M. Hashim, M. Dirar, and A. KhalafAllah. "String Model Solution for Linearized GSR Quantum Theory." International Journal of Multidisciplinary Research (IJMR) (2018).*
14. *Kaya, Doğan, and Salah M. El-Sayed. "A numerical solution of the Klein–Gordon equation and convergence of the decomposition method." Applied mathematics and computation 156, no. 2 (2004): 341-353.*