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TOTAL COMPLEMENTARY TREE DOMINATION NUMBER OF GRAPHS

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ABSTRACT
Let G = (V, E) be a non-trivial, simple, finite and undirected graph. A dominating set D is called a complementary tree dominating set if the induced subgraph <V–D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. A dominating set D is called a total complementary tree dominating set (tctd-set) if every vertex $v \in V$ is adjacent to an element of D and <V–D> is a tree. The minimum cardinality of a total complementary tree dominating set is called the total complementary tree domination number of G and is denoted by $\gamma_{tctd}(G)$. In this paper, bounds for $\gamma_{tctd}(G)$ and its exact values for particular classes of graphs are found. Some results on total complementary tree domination numbers are also established.

KEYWORDS: *Total domination, total complementary tree domination.*
AMS Subject Classification (2010): 05C69.

1 INTRODUCTION

The graphs considered here are nontrivial, simple, finite and undirected. Let G be a graph with vertex set V(G) and edge set E(G). For $v \in V(G)$ the neighbourhood N(v) of v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. $N_i[v] = \{v \in V(G) : d(u, v) = i\}$ is called the ith neighbourhood of v. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 defined as the graph G of order p_1 and p_1 copies of G_2 and then joining the ith copy of G_2. It has $p_1 (1 + p_2)$ vertices and $q_1 + p_1q_2 + p_1p_2$ edges. For any graph G, the corona $G \odot K_1$ is denoted by G^+. $C_3^+ - v$, where v is a pendant vertex of C_3^+ is called a bull graph. The concept of domination was first studied by Ore [5]. A set $D \subseteq V$ is said to a dominating set of G, if every vertex in V–D is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. The concept of complementary tree domination was introduced by S. Muthammai, M. Bhanumathi and P. Vidhya in [4]. A dominating set $D \subseteq V$ is called a complementary tree dominating (ctd) set, if the induced subgraph <V–D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. A dominating set D is called a total dominating set if every vertex $v \in V$ is adjacent to an element of D. The minimum cardinality of a total dominating set in G is denoted by $\gamma_t(G)$. A dominating set D is called a total complementary tree dominating set of every vertex $v \in V$ is adjacent to an element of D and <V–D> is a tree. The minimum cardinality of a total complementary tree dominating set is called the total complementary tree domination number of G and is denoted by $\gamma_{tctd}(G)$.

Kulli and Janakiram [7] introduced the concept of split domination in graphs. A dominating set D of a graph G = (V, E) is a split dominating set if the induced subgraph <V–D> is disconnected. The split domination number $\gamma_s(G)$ of a graph G is the minimum cardinality of a split dominating set. Kulli and Janakiram [8] introduced the concept of non-split domination in graphs. A dominating set D of a graph G = (V, E) is a non split dominating set if the induced subgraph <V−D> is connected. The non split domination number $\gamma_{ns}(G)$ of a graph G is the minimum cardinality of a non split dominating set. In this paper bounds for $\gamma_{tctd}(G)$ and its exact values for particular classes of graphs are found. Some results on total complementary tree domination number are also established.

2 PRIOR RESULTS

Theorem 2.1. [5]

A dominating set D of a graph G = (V, E) is a minimal dominating set if and only if for each vertex v in D, one of the following two conditions hold

(i) v is an isolatex vertex of D

(ii) there exist a vertex u in V−D for which $N(u) \cap D = \{v\}$

Theorem 2.2. [1]

(i) if G is a connected graph with $p \geq 3$ vertices, then $\gamma_t(G) = 2p/3$

(ii) if G has p vertices and no isolated then $\gamma_t(G) = p - \Delta(G) + 1$

(iii) if G is connected $\Delta(G) \leq p - 1$, then $\gamma_t(G) = p - \Delta(G)$

Observation 2.1. [4]

(i) For any connected graph, $\gamma(G) \leq \gamma_{ctd}(G)$.

(ii) For any connected graph H of G, $\gamma_{ctd}(G) \leq \gamma_{ctd}(H)$.

(iii) For any connected graph G with $p \geq 2$, $\gamma_{ctd}(G) \leq p-1$.

3 TOTAL COMPLEMENTARY TREE DOMINATION NUMBER OF GRAPHS

Definition 3.1.

A complementary tree dominating set $D \subseteq V$ of a connected graph G = (V, E) is said to be a total complementary tree dominating set (tctd-set), if the induced subgraph <D> has no isolated vertices.

The minimum cardinality of a tctd-set D of a connected graph G is called the total complementary tree domination number, denoted by $\gamma_{tctd}(G)$ and such a set D is called a γ_{tctd}-set.

A total ctd-set D of G is minimal, if no proper subset of D is a tctd-set of G. It is to be noted that γ_{tctd}-set exists for all connected graphs.

Observation 3.1.

Since every total complementary tree dominating set is a complementary tree dominating set, $\gamma_{ctd}(G) \leq \gamma_{tctd}(G)$ for any connected graph G. Also, every total complementary tree dominating set is a total dominating set. Therefore

$\gamma_t(G) \leq \gamma_{tctd}(G)$ for any connected graph G.

Note.

If G is a connected graph and H is any connected spanning (induced) subgraph of G, then it is not necessary that the inequality $\gamma_{tctd}(G) \leq \gamma_{tctd}(H)$ holds.

Example 3.1.

For the graph G in Figure 1, H_1 is a spanning subgraph of G and H_2 is an induced subgraph of G.

$\gamma_{tctd}(G) = 3$, whereas $\gamma_{tctd}(H_1) = 2$ and $\gamma_{tctd}(H_2) = 2$.

In analogous to Theorem 2.4 [4], following result characterizes minimal total complementary tree dominating sets and is stated without proof.

Theorem 3.1.

A total complementary dominating set $D \subseteq V$ of a connected graph G = (V, E) is minimal if and only if for each vertex $v \in D$, one of the following conditions hold

(i) v is not an isolated vertex of G.

(ii) There exists a vertex u in V−D such that $N(u) \cap D = \{v\}$

(iii) $N(v) \cup (V - D) = \phi$

(iv) The subgraph $\langle (V - D) \cup \{v\} \rangle$ of G either contains a cycle or disconnected.

(v) $D - \{v\}$ contains isolated vertices.

Observation 3.2.

(i) For any connected graph G with atleast three vertices, $2 \leq \gamma_{tctd}(G) \leq p-1$. The lower bound is attained, when $G \cong W_p$, wheel on p vertices and the upper bound is attained, when $G \cong K_{1,p}$, $p \geq 3$.

(ii) If $\gamma_{tctd}(G) \leq p-2$, then pendant vertices and supports of G are members of every tctd-set and hence, $\gamma_{tctd}(G) \geq m+n$, where m and n are number of pendant vertices and supports of G, respectively.

Observation 3.3.

(i) For the path P_n, $\gamma_{tctd}(P_n) = n-2$, $n \geq 4$.

(ii) For the cycle C_n, $\gamma_{tctd}(C_n) = n-2$, $n \geq 4$.
(iii) For the complete graph K_n, $\gamma_{tctd}(K_n) = n-2$, $n \geq 4$.
(iv) For the star $K_{1,n}$, $\gamma_{tctd}(K_{1,n}) = n$, $n \geq 3$.
(v) For the complete bipartite graph $K_{m,n}$, $\gamma_{tctd}(K_{m,n}) = \min(m, n)$, $m, n \geq 2$.
(vi) $\gamma_{tctd}(C_n \circ K_1) = 2n-1$, $n \geq 3$.
Here, $V(C_n \circ K_1)$ - a pendant vetex forms a γ_{tctd}-set.
(vii) For the wheel W_n with n vertices, $\gamma_{tctd}(W_n) = 2$, $n \geq 4$.
(viii) For the subdivision graph of star $K_{1,n}$, $\gamma_{tctd}(G) = 2n$, $n \geq 2$.
Here, all the n pendant vertices and n support vertices forms a γ_{tctd}-set.

Proposition 3.1.

Let $C_n^{(t)}$, $t \geq 2$ be the one point union of t cycles of length n ($n \geq 3$), then

$$\gamma_{tctd}(C_n^{(t)}) = \begin{cases} (n-1)t, & n = 3 \\ (n-2)t+1, & n = 4 \\ (n-3)t+1, & n \geq 5. \end{cases}$$

Proof.

$G = C_n^{(t)}$ and u be the point of union of t cycles of length n.

G has t(n–1)+1 vertices. Let the vertex set of k^{th} cycle in $C_n^{(t)}$ be $V_k = \{u, u_{k1}, u_{k2}, ..., u_{k,n-1}\}$, $k = 1, 2, ..., t$.

Case 1. n = 3.

Let $D_k = \{u_{k1}, u_{k2}\}$, $k = 1, 2, ..., t$ and $D = \bigcup_{k=1}^{t} D_k \subseteq V(G)$.

Then, $\langle V-D\rangle \cong K_1$ and let $v \in D$, then $\langle V-(D-\{v\})\rangle$ either contains a cycle or is disconnected and hence, D is a minimum tctd-set of G and $\gamma_{tctd}(G) = |D| = (n-1)t$.

Case 2. n = 4.

Let $D_k = \{u_{k2}, u_{k3}\}$, $k = 1, 2, ..., t$ and $D = \bigcup_{k=1}^{t} D_k \cup \{u_{11}\} \subseteq V(G)$.

Then, $\langle V-D\rangle \cong K_{1,t}$. As in case 1, D is a minimum tctd-set of G and hence, $\gamma_{tctd}(G) = |D| = (n-2)t+1$.

Case 3. $n \geq 5$.

Let $D_k = \{u_{k2}, u_{k3}, ..., u_{k,n-2}\}$, $k = 1, 2, ..., t$ and $D = \bigcup_{k=1}^{t} D_k \cup \{u_{11}\} \subseteq V(G)$.

Then, $\langle V-D\rangle \cong K_{1,2t-1}$. As in case 1, D is a minimum tctd-set of G and hence $\gamma_{tctd}(G) = |D| = (n-3)t+1$. □

4 BOUNDS AND SOME EXACT VALUES FOR THE TOTAL COMPLEMENTARY TREE DOMINATION NUMBER

In the following, a lower bound of $\gamma_{tctd}(G)$ in terms of order and size of the graph G is given.

Theorem 4.1.

For any connected (p, q) ($p \geq 3$) graph G,

$$\gamma_{tctd}(G) \geq \left\lfloor \frac{2(2p-q-1)}{3} \right\rfloor$$

Proof.

Let D be a γ_{tctd}-set of G. Let t be the number of edges in G having one vertex in D and the other in V–D and s be the number of edges in D. The number of vertices in $\langle V-D\rangle$ is $p-\gamma_{tctd}(G)$ and since $\langle V-D\rangle$ is a tree, number of edges in $\langle V-D\rangle$ is $p-\gamma_{tctd}(G)-1$. Since there are atleast $p-\gamma_{tctd}(G)$ edges from V–D to D, $t \geq p-\gamma_{tctd}(G)$. Also, $\sum_{v \in D} \deg_{\langle D\rangle}(v) = 2s$ and $\deg_{\langle D\rangle}(v) \geq 1$, for each $v \in D$ implies that $2s \geq \gamma_{tctd}(G)$. Hence, $s \geq \frac{\gamma_{tctd}(G)}{2}$.

Therefore, q = number of edges in $\langle D\rangle + t + p - \gamma_{tctd}(G) - 1$

$$\geq \frac{\gamma_{tctd}(G)}{2} + p - \gamma_{tctd}(G) + p - \gamma_{tctd}(G) - 1$$

That is, $q \geq 2p - 1 - \frac{3\gamma_{tctd}(G)}{2}$.

Hence, $\gamma_{ctd}(G) \geq \left\lfloor \frac{2(2p-q-1)}{3} \right\rfloor$.

This bound is attained, if $G \cong C_4$. □

Corollary 4.1.

If G is a tree on p vertices, then $\gamma_{tctd}(G) \geq \frac{2p}{3}$ and is attained, if G is the graph obtained from P_m^+ $(m \geq 2)$ by subdividing each pendant edge exactly once.

Proof.

Replacing q by p–1 in Theorem 4.1 $\gamma_{tctd}(G) \geq \frac{2p}{3}$ is obtained. □

Observation 4.1.

Since $\gamma_t(G) \geq \left\lceil \frac{P}{\Delta(G)} \right\rceil$ for a connected graph G and $\gamma_t(G) \geq \gamma_{tctd}(G)$, we have $\left\lceil \frac{P}{\Delta(G)} \right\rceil \leq \gamma_t(G)$. This bound is attained, if $G \cong C_4, C_5, W_n, n \geq 4$.

Theorem 4.2.

Let G be a connected graph with $\delta(G) \geq 2$ and $diam(G) \geq 3$. If there exists a vertex $v \in V(G)$ such that the induced subgraph <N(v)> is totally disconnected, then $\gamma_{tctd}(G) \leq p - \delta(G)$, where N(v) is the neighbourhood set of v.

Proof.

Let $v \in V(G)$ be such that <N(v)> is totally disconnected. Then, $<N[v]> \cong K_{1,t}$ where $t = deg(v) \geq \delta(G)$. Let $u \in N(v)$. Then, $D = V - N[v] \cup \{u\}$ is a total dominating of G. Also, $<V - D> \cong K_{1,t-1}$.
Therefore, D is a tctd-set of G and hence

$$\begin{aligned}\gamma_{tctd}(G) &\leq |D| \\ &= |V - N[v] \cup \{u\}| \\ &= p - (t+1) + 1 \\ &= p - t \\ &\leq p - \delta(G)\end{aligned}$$

Equality holds, if $G \cong C_n, n \geq 6$. □

Remark 4.1.

Let G be a connected graph with $diam(G) = 2$ and $\delta(G) \geq 2$. If there exists a vertex $v \in V(G)$ such that <N(v)> is totally disconnected and $<N_2(v)>$ contains no isolated vertices, then

$$\gamma_{tctd}(G) \leq p - \delta(G),$$

where $N_2(v)$ is the second neighbourhood set of v.

Theorem 4.3.

Let G be a connected graph with $diam(G) = 2$. If there exists a vertex $v \in V(G)$ such that $<N_2(v)>$ is a tree, then $\gamma_{tctd}(G) \leq \Delta(G) + 1$.

Proof.

Let $v \in V(G)$ be such that $<N_2(v)>$ is a tree.
Since diam(G) = 2, N(v) is a dominating set of G.
Therefore, N[v] is a total dominating set of G.
Since $<N_2(v)>$ is a tree, N[v] is a tctd-set of G. Hence,

$$\begin{aligned}\gamma_{tctd}(G) &\leq |N[v]| \\ &= deg_G(v) + 1 \\ &\leq \Delta(G) + 1\end{aligned}$$

This bound is attained, if $G \cong C_5$. □

Theorem 4.4.

Let T be a tree with atleast three vertices. Then the set of all pendant vertices and supports of T are tctd-set if and only if

(i) each nonsupport of T of degree atleast 2 is adjacent to exactly one support and

(ii) no two nonsupports of degree atleast 2 is adjacent to the same support.

Proof.

Let D be the set of all pendant vertices and supports of T and be a tctd-set of T. Then, <V−D> is a tree and it contains nonsupports of T.

If the above conditions do not hold, then T contains a cycle.

Conversely, if the condition (i) and (ii) hold, then V(T) − nonsupports is a tctd-set of G. □

Remark 4.2.

From the above theorem, T is the tree obtained from P_n^+ (n ≥ 2) by subdividing each pendant edge exactly once.

Theorem 4.5.

Let G be a connected graph with atleast four vertices, then $\gamma_{tctd}(G) = 2$ if and only if G is one of the following graphs.

(i) G is the graph obtained from $K_1 + T$ with one pendant edge attached at the vertex of K_1, where T is any tree with atleast two vertices.

(ii) G is the graph obtained from a tree by joining each of the vertices of the tree to atleast one of the vertices of K_2 such that $\deg_G v \geq 2$, for all $v \in V(K_2)$.

Proof.

Let G be one of the graph mentioned in (i) and (ii). Since G is not isomorphic to $K_1 + T$, for any tree T, $\gamma_{tctd}(G) \geq 2$.

If G is the graph as in (i), the subset of V(G) consisting of the vertex of K_1 and the pendant vertex of G forms a tctd-set of G.

Therefore, $\gamma_{tctd}(G) \leq 2$ and hence $\gamma_{tctd}(G) = 2$. Conversely, assume $\gamma_{tctd}(G) = 2$. Then, there exists a tctd-set D such that |D| = 2.

Let D = {u, v}.

(i) If u or v is a pendant vertex in G, then all the vertices of V−D are adjacent to v or u. Therefore, G is the graph mentioned in (i).

(ii) Let $\deg_G(u) \geq 2$ and $\deg_G(v) \geq 2$. Since <V−D> is a tree and D is a total dominating set of G, each vertex in V−D is adjacent to atleast one vertex in D. Hence, G is the graph as in (ii).

□

Theorem 4.6.

Let G be a connected (p, q) graph with p ≥ 3 and δ(G) = 1. Then, $\gamma_{tctd}(G) = p-1$ if and only if either

(i) every vertex of degree atleast 2 is a support (or)

(ii) the subgraph of G induced by nonsupports of G of degree atleast 2 is either totally disconnected (or) contains exactly one vertex

Proof.

Let G be a connected graph with p ≥ 3 and δ(G) = 1

Assume $\gamma_{tctd}(G) = p - 1$

Let D be a tctd-set of G such that |D| = p − 1. Then V−D contains exactly one vertex of G. Let S be the set of all pendant vertices and supports of G. Then, S ⊆ D.

If S = D, then the vertex in V−D is neither a pendant vertex nor a support of G and is adjacent to atleast two supports of G. That is, subgraph of G induced by the vertices of degree atleast 2 and are not the supports contains exactly one vertex.

If S = V(G), then since D contains (p−1) vertices, one pendant vertex must be in V−D.

In this case, every vertex of degree atleast 2 is a support of G.

Let v ∈ V−D, then v is adjacent to atleast one vertex, say w in D.

If w ∈ D−S and is adjacent to a vertex in D−S, then D−{w} is a tctd-set of G. Therefore, w is adjacent to a vertex in S. That is, w is adjacent to atleast one support of G.

Hence, vertices in D-S are independent. That is, the vertices of G, which are neither pendant vertices nor supports, are independent in G.

Conversely, if every vertex of degree atleast 2 in G is a support, then V − {a pendant vertex} is a tctd-set of G and no vertex in D can be included in V−D and hence $\gamma_{tctd}(G) = p-1$.

Let the subgraph, say U of G induced by nonsupport vertices of degree atleast 2 either totally connected or contains exactly one vertex, then V − {u}, where u ∈ U is a tctd-set of G and hence, $\gamma_{tctd}(G) = p-1$. □

Theorem 4.7.

Let G be a connected graph with $p \geq 4$. If there exists an induced path P of length 2 in G such that central vertex of P has degree atleast 3 and none of the vertices of P are supports and $<V(G) - V(P)>$ has no isolated vertices, then $\gamma_{tctd}(G) \leq p-3$.

Proof.

Let $D = V(G) - V(P)$.
Since central vertex of P has degree atleast 3, each vertex in P is adjacent to atleast one vertex in D.
Also $<V - D> = <V(P)> \cong P_3$.
Hence, D is a ctd-set of G.
Since $<D>$ has no isolated vertices, D is a total ctd-set of G.
Therefore, $\gamma_{tctd}(G) \leq |V(G) - V(P)| = p-3$. □

Theorem 4.8.

Let G be a connected graph with atleast four vertices and let D be a γ_t-set of G such that $<V-D>$ is complete or $<V-D> \cong mK_2$, $m \geq 1$. Then $\gamma_{tctd}(G) = p-2$.

Proof.

Let D be a γ_t-set of G such that $<V-D>$ is complete. If $<V-D> \cong K_2$, then D itself is a γ_{tctd}-set of G and hence $\gamma_{tctd}(G) = \gamma_t(G) = p-2$.
Let $<V-D> \cong K_m$, $m \geq 3$. Then, $D \cup V(K_{m-2})$ is a tctd-set of G. Similarly, if $<V-D> \cong mK_2$, $m \geq 2$, then $D \cup V((m-1)K_2)$ is a tctd-set of G. In both the cases, $\gamma_{tctd}(G) \leq p-2$. Also, since $<V-D>$ is a tree, no subset of V(G) containing atmost (p–3) vertices is a tctd-set of G and hence, $\gamma_{tctd}(G) = p-2$. □

Theorem 4.9.

Let G be a connected graph with atleast three vertices, then $\gamma_{tctd}(G) = p-2$ if and only if

(i) $G \cong K_p$, $p \geq 4$
(ii) $G \cong C_p$, $p \geq 3$, P_p, $p \geq 6$
(iii) G has atleast one of the following
 (a) If G has an induced path of length 2 in G, then the central vertex is of degree 2 in G.
 (b) If G has an induced path P of length 2 in G and if the central vertex of P is of degree atleast three in G, then either central vertex of P is a support of G or atleast one of the pendant vertices of P is a support or a pendant vertex of G such that either G has atleast two adjacent nonsupport vertices of degree atleast 2 (or) $V(G) - V(P)$ has isolated vertices.

Proof.

Let G be a connected graph with $\gamma_{tctd}(G) = p-2$
Let D be a tctd-set of G such that $|D| = p-2$. Then $<V-D> \cong K_2$.
By the Theorem 4.7, if there exists an induced path P of length 2 in G such that
(a) the central vertex of P has degree atleast three in G
(b) None of the vertices of P are supports of G, and
(c) $<V(G) - V(P))$ has no isolated vertices, then $\gamma_{tctd}(G) \leq p-3$.
 Hence, atleast one of the following holds
 (i) There exists no induced path of length 2 in G
 (ii) The central vertex of induced path of length 2 in G is of degree 2 in G
 (iii) If the central vertex of an induced path P of length 2 in G is of degree atleast three, then either
  (a) central vertex of P is a support (or) atleast one of the pendant vertices of P is a support or a pendant vertex of G (or)
  (b) $<V(G) - V(P)>$ has atleast one isolated vertex.

If (i) holds, then any two vertices of G are adjacent and hence $G \cong K_p$, $p \geq 3$. If the central vertex of each induced path of length 2 in G is of degree 2 in G, then $G \cong P_p$ (or) C_p, $p \geq 3$. But, if $G \cong P_p$, $p = 3, 4, 5$, $\gamma_{tctd}(G) = p-1$. Hence, $G \cong P_p$, $p \geq 6$. Let the central vertex of each induced path of length 2 in G is of degree atleast 3 in G. If each vertex of G of degree atleast 2 is a support, then $\gamma_{tctd}(G) = p-1$. (by Theorem 4.6). Similarly, if the subgraph of G induced by nonsupports of G of degree atleast 2 is either totally disconnected or contains exactly one vertex, then $\gamma_{tctd}(G) = p-1$.
Hence, if the central vertex of an induced path P of length 2 in G is of degree atleast three in G, then either central vertex of P is a support of G or atleast one of the pendant vertices of P is a support of G or a pendant vertex of G such that G has atleast two adjacent nonsupport vertices of degree atleast 2 (or) $V(G) - V(P)$ has isolated vertices. Therefore, G is one of the graphs given in (i), (ii) and (iii).
Conversely, let G be one of the graphs given in (i), (ii) and (iii). If $G \cong K_p$, $p \geq 4$, P_p, $p \geq 6$, C_p, $p \geq 3$, then $\gamma_{tctd}(G) = p-2$. If G is the graph satisfying (iii), then every tctd-set of G contains both supports and pendant vertices, all the vertices of G except two adjacent vertices,

which are nonsupports of degree atleast 2, are to be included in the tctd-set and hence, $\gamma_{tctd}(G) \leq p-2$. □

Theorem 4.10.

Let G be a connected graph with $\delta(G) = 1$ and let $S \subseteq V(G)$ be the set consisting of supports and pendants vertices of G. If <V−S> is a tree and each vertex in V−S is adjacent to a support in G, then $\gamma_{tctd}(G) = m+n$.

Proof.

Since S has no isolated vertices, S is a tctd-set of G. Therefore, $\gamma_{tctd}(G) \leq |S| = m+n$. Also, $\gamma_{tctd}(G) \geq m+n$ and hence, $\gamma_{tctd}(G) = m+n$. □

5 Relationship between Total Complementary Tree Domination Number and other Parameters

In this section, the relationship between $\gamma_{tctd}(G)$ and $\gamma_t(G)$, $\gamma_{tns}(G)$, $\gamma_s(G)$ are found.

Theorem 5.1.

Let G be a connected graph. If $\kappa(G) > \gamma_t(G)$ and if there exists a γ_t-set D of G such that <V−D> is acyclic, then $\gamma_{tctd}(G) = \gamma_t(G)$.

Proof.

Let D be a γ_t-set of G. Since $\kappa(G) > \gamma_t(G)$, <V−D> is connected and since <V−D> is acyclic, and is a tree.

Therefore, D is a tctd-set of G and $\gamma_{tctd}(G) \leq |D| = \gamma_t(G)$.

But, $\gamma_t(G) \leq \gamma_{tctd}(G)$.

Therefore, $\gamma_{tctd} = \gamma_t(G)$. □

Observation 5.1.

Every connected graph contains a spanning connected subgraph H such that $\gamma_{tctd}(H) = \gamma_{tns}(G)$, where $\gamma_{tns}(G)$ is the minimum cardinality of a nonsplit dominating set having no isolated vertices.

Theorem 5.2.

Let G be a connected graph and let D be a tctd-set of G. If there exists a vertex $v \in D$ such that $N(v) \subseteq D$, then $\gamma_s(G) < \gamma_{tctd}(G)$, where $\gamma_s(G)$ is the split domination number of G.

Proof.

Let D be a tctd-set of G. Therefore, <V−D> is a tree and $|D| \leq \gamma_{tctd}(G)$. Let $v \in D$ be such that $N(v) \subseteq D$, then $D - \{v\}$ is a split dominating set of G, since $V - [D - \{v\}]$ is disconnected with an isolated vertex.

Hence, $\gamma_s(G) \leq |D - \{v\}| \leq \gamma_{tctd}(G) + 1$.

Therefore, $\gamma_s(G) < \gamma_{tctd}(G)$. □

In the following, Nordhaus-Gaddum type result for total complementary tree domination number is established.

Theorem 5.3.

Let G be a graph such that both G and its complement $\overline{G}$ are connected. Then

$$4 \leq \gamma_{tctd}(G) + \gamma_{tctd}(\overline{G}) \leq 2(p-1)$$

$$4 \leq \gamma_{tctd}(G).\gamma_{tctd}(\overline{G}) \leq (p-1)^2$$

The upper bound is attained, if G ≅ Bull graph and the lower bound is attained, if G is the cycle C_4 with one pendant edge attached at a vertex of C_4.

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