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MEEKLY π-NORMAL SPACES IN GENERAL TOPOLOGY

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ABSTRACT

In this paper, A new generalization of normality called meekly π-normality is introduced and studied which is a simultaneous generalization of π-normality and β-normality. Interrelation among some existing variants of normal spaces is discussed and characterizations of meekly π-normal space with some existing variants of normal spaces are obtained.

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KEYWORDS : ***β-normal, π-normal, almost normal, meekly normal, softly regular, κ-normal, κβ-normal.***

1. INTRODUCTION

Normality plays a prominent role in general topology. In 1968, Zaitsev [**25**] introduced the notion of quasi normality is a weaker form of normality and obtained its properties. In 1970, the concept of almost normality was introduced by Singal and Arya [**16**]. In 1973, the notion of mild normality was introduced by Shchepin [**20**] and, Singal and Singal [**17**] independently. In 2011, Arhangel'skii and Ludwig [**1**] introduced the concept of α-normal and β-normal spaces and obtained their p[roperties. Eva Murtinovain [**15**] provided an example of a β-normal Tychonoff space which is not normal. In 2002, Kohli and Das [**10**] introduced θ-normal topological spaces and obtained their characterizations. In 2008, Kalantan [**9**] introduced π-normal topological spaces and obtained their characterizations. In 2015, Sharma and Kumar [**22**] introduced a new class of normal spaces called softly normal and obtained a characterization of softly normal space. In 2018, Kumar and Sharma [**12**] introduced the concepts of softly regular and partly regular spaces and obtained some characterizations of softly regular spaces. In 2023, Kumar [**13**] introduced the concepts of epi π-normal spaces, which lies between epi-normal and epi-almost normal spaces, and epi-normal and epi-quasi normal spaces. Interrelation among some existing variants of normal spaces is discussed and characterizations of epi π-normal space with some existing variants of normal spaces are obtained.

2. PRELIMINARIES

Let X be a topological space and let $A \subset X$. Throughout the present paper the **closure** of a set A will be denoted by cl(A) and the **interior** by int(A). A set $U \subset X$ is said to be **regularly open** [**14**] if U = int(cl(U)). The complement of a regularly open set is called **regularly closed**. The finite union of regular open sets is said to be π-**open [25]**. The complement of a π-open set is said to be **π-closed.** A topological space is said to be **normal** [**3, 7, 8**] if for any pair of disjoint-closed subsets A and B of X can be separated. A space is **k-normal** [**20**] (**mildly normal** [**17**]) if for every pair of disjoint regularly closed sets E, F of X there exist disjoint open subsets U and V of X such that $E \subset U$ and $F \subset V$. A topological space is said to be **almost normal** [**16**] if for every pair of disjoint closed sets A and B one of which is regularly closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A topological space is said to be **π-normal** [**9**] if for every pair of disjoint closed sets A and B, one of which is π-closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A topological space X is said to be **almost regular** [**16**] if for every regularly closed set A and a point $x \notin A$, there exist disjoint open sets U and V such that $A \subset U$ and $x \in V$. A topological space is said to be **softly regular** [**12**] if for every π-closed set A and a point $x \notin A$, there exist two open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$. A topological space X is said to be **α-normal** [**1**] if for any two disjoint closed subsets A and B of X, there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap U$ is dense in B. A space X is **β-normal** [**1**] if for any two disjoint closed subsets A and B of X, there exist open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap U$ is dense in B, and $cl(U) \cap cl(V) = \phi$. A topological space is called **almost β-normal** [**5**] if for every pair of disjoint closed sets A and B, one of which is regularly closed, there exist disjoint open sets U and V such that $cl(U \cap A) = A$, $cl(V \cap B) = B$, and $cl(U) \cap cl(V) = \phi$. A topological space X is said to

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be **βk-normal** [**19**] if for every pair of disjoint regularly closed subsets A and B of X, there exist disjoint open sets U and V of X such that cl(A ∩ U) = A, cl(B ∩ U) = B and cl(U) ∩ cl(V) = ϕ. A space X is said to be **semi-normal** if for every closed set A contained in an open set U, there exists a regularly open set V such that A ⊂ V ⊂ U.

3. MEEKLY π-NORMAL

Definition 3.1. A topological space is called **meekly π-normal** if for every pair of disjoint closed sets A and B, one of which is π-closed, there exist disjoint open sets U and V such that cl(U ∩ A) = A, cl(V ∩ B) = B, and cl(U) ∩ cl(V) = ϕ.

From the definitions it is obvious that every normal space is π-normal and every π-normal space is meekly π-normal.

Theorem 3.2. Every π-normal space is meekly π-normal.
Proof. Let X be a π-normal space. Let A and B be two disjoint closed sets in X, one of which (say A) is π-closed. Since X is π-normal there exist disjoint open sets W and V containing A and B respectively. Since W ∩ V = ϕ, W ∩ cl(V) = ϕ. Let U = int(A). Then cl(U) ∩ cl(V) = ϕ, cl(U ∩ A) = A, and cl(V ∩ B) = B. So, the space is meekly π-normal.

The following implications hold but none are reversible.

normal	⇒	π-normal	⇒	almost normal	⇒	k-normal
⇓		⇓		⇓		⇓
β-normal	⇒	meekly π-normal	⇒	almost β-normal	⇒	βk-normal

Example 3.3. Let X = {a, b, c, d} and ℑ = {ϕ, X, {b}, {c}, {c, d}, {b, c}, {a, b, c}, {b, c, d}}. Then the space (X, ℑ) is not meekly π-normal since for π-closed A = {a, b} and disjoint closed set B = {d}, there does not exist two open sets U and V such that cl(U ∩ A) = A, cl(B ∩ V) = B, and U ∩ V = ϕ.

Example 3.4. Let X be the union of any infinite set Y and two distinct one point sets p and q. The modified Fort space on X as defined in [**23**] is almost β-normal as well as kβ-normal but not β-normal. In X any subset of Y is open and any set containing p or q open if and only if it contains all but a finite number of points in Y. This space is not β-normal even not α-normal [**1**] because for disjoint closed sets {p} and {q} there does not exist two disjoint open sets separating them. The regularly closed sets of this space are finite subsets of Y and sets of the form A ∪ {p, q}, where A ⊂ Y is infinite. Thus the space is almost β-normal.

Arhangel'skii and Ludwig [**1**] have shown that a space is normal if and only if it is κ-normal and β-normal. Therefore, every non-normal space which is almost normal is an example of a κ-normal, almost β-normal space which is not β-normal.

Recall that a Hausdorff space X is said to be **extremally disconnected** if the closure of every open set in X is open.

A point x ∈ X is called a **θ-limit point** [**24**] of A if every closed neighbourhood of x intersects A. Let $cl_\theta(A)$ denotes the set of all θ-limit points of A. The set A is called θ-closed if $A = cl_\theta(A)$.

Definition 3.5. A topological space X is said to be
(i) θ-normal [**10**] if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets;
(ii) Weakly θ-normal (wθ-normal) [**10**] if every pair of disjoint θ-closed sets are contained in disjoint open sets.

Theorem 3.6. Every extremally disconnected meekly π-normal space is π-normal.
Proof. Let X be an extremally disconnected meekly π-normal space and let A be a π-closed set disjoint from the closed set B. By meekly π-normality of X, there exist disjoint open sets U and V such that cl(U ∩ A) = A, cl(V ∩ B) = B and cl(U) ∩ cl(V) = ϕ. Thus A ⊂ cl(U) and B ⊂ cl(V). By the extremally disconnectedness of X, cl(U) and cl(V) are disjoint open sets containing A and B respectively.

Theorem 3.7. Every T_1 almost β-normal space is almost regular [**5**].

Theorem 3.8. Every T_1 meekly π-normal space is softly regular.
Proof. Let A be a π-closed set in X and x be a point outside A. Since X is a T_1-space and every singleton is closed in a T_1-space, by meekly π-normality there exist disjoint open sets U and V such that $x \in U$, $cl(V \cap A) = A$, $cl(U) \cap cl(V) = \phi$. Since $A \subset cl(V)$, U and $X - cl(U)$ are disjoint open sets containing {x} and A, respectively. Thus, the space is softly regular.

Corollary 3.9. Every T_1 meekly π-normal space is almost regular.
Proof. Since every softly regular space is almost regular, so proof is easy.

Theorem 3.10. An almost regular weakly θ-normal space is mildly normal space [**11**].

Corollary 3.11. A softly regular weakly θ-normal space is mildly normal space
Proof. Since every softly regular space is almost regular, so proof is easy.

Corollary 3.12. In a T_1-space, weak θ-normality and meekly π-normality implies mildly normality.
Proof. Let X be a T_1 weakly θ-normal, meekly π-normal space. Then by Corollaty 3.9, X is almost regular. Since every softly regular weakly θ-normal space is κ-normal, so X is κ-normal.

Corollary 3.13. In the class of T_1, θ-normal spaces, every meekly π-normal space is π-normal.
Proof. Let X be a T_1 space which is θ-normal as well as meekly π-normal. Since every T_1 meekly π-normal space is softly regular, so X is π-normal.

Corollary 3.14. In the class of T_1, paracompact spaces, every meekly π-normal space is π-normal.
Proof. Since every paracompact space is θ-normal [**10**], the result holds by Corollary 3.13.

Recall that a space X is said to be **almost compact** [**4**] if every open cover of X has a finite subcollection, the closure of whose members covers X.

Corollary 3.15. An almost compact, meekly π-normal, T_1-space is κ-normal.
Proof. The proof is immediate from the result Theorem 3.8 of Singal and Singal [**17**] and since every T_1 meekly π-normal space is almost regular that an almost regular almost compact space is κ-normal.

Corollary 3.16. A Lindel¨of, meekly π-normal, T_1-space is κ-normal.
Proof. Since an almost regular Lindel¨of space is κ-normal [**17**], and since every T_1 meekly π-normal space is almost regular, the proof is immediate

Remark 3.17. The T_1 axiom in the above theorem cannot be relaxed since there exist meekly π-normal spaces which are not almost regular.

Example 3.18. Let X = {a, b, c} and $\Im$ = {{a}, {c}, {a, c}, ϕ, X}. Then X is vacuously normal, thus meekly π-normal but not almost regular as the regularly closed set {a, b} and any point outside it cannot be separated by disjoint open sets.

Theorem 3.19. In the class of T_1, semi-normal spaces, every meekly π-normal space is regular.
Proof. Let X be a T_1, semi-normal, and meekly π-normal space. Let A be a closed subset of X and $x \notin A$. Since X is a T_1-space, the singlton set {x} is closed. So by semi-normality of X, there exists a regularly open set U such that $\{x\} \subset U \subset X - A$. Here $F = X - U$ is a regularly closed set containing A with $x \notin F$. As X is a meekly π-normal T_1-space, X is softly regular by Theorem 3.7. Thus there exist disjoint open sets V and W such that $x \in V$ and $A \subset F \subset W$. Hence X is regular.

The following theorem provides a characterization of meekly π-normality.

Theorem 3.20. For any topological space X, the following are equivalent:
(1). X is meekly π-normal;

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(2). whenever $E, F \subset X$ are disjoint closed sets and E is π-closed, there is an open set V such that $F = cl(V \cap F)$ and $E \cap cl(V) = \phi$;
(3). whenever $E \subset X$ is closed, $U \subset X$ is π-open, and $E \subset U$, there is an open set V such that $E = cl(E \cap V) \subset cl(V) \subset U$.
Proof. [(1) ⇒ (2)]. Suppose that $E, F \subset X$ are disjoint closed sets and E is π-closed. Since X is meekly π-normal, there exist open sets U and V such that $E = cl(U \cap E) \subset cl(U)$, $F = cl(V \cap F) \subset cl(V)$, and $cl(U) \cap cl(V) = \phi$. Then $E \cap cl(V) = \phi$.

[(2) ⇒ (1)]. Suppose that $E, F \subset X$ are disjoint closed sets and E is π-closed. By the assumption, there exists an open set V such that $F = cl(V \cap F)$ and $E \cap cl(V) = \phi$. Let $U = int(E)$. Then $E = cl(U \cap E)$ and $cl(U) \cap cl(V) = E \cap clV) = \phi$.

[(1) ⇒ (3)]. Suppose that E is closed, U is π-open, and $E \subset U$. Since U is π-open, $X - U$ is π-closed. Since X is meekly π-normal, there are open sets O and V such that $X - U = cl(O \cap (X - U) \subset O$, $E = cl(V \cap E) \subset cl(V)$, and $cl(O) \cap cl(V) = \phi$. Then $(X - U) \cap cl(V) = \phi$ which means that $cl(V) \subset U$.

[(3) ⇒ (2)]. Suppose that $E, F \subset X$ are disjoint closed sets and E is π-closed. Then $F \subset X - E$ and $X - E$ is π-open. By the hypothesis, there is an open set V such that $F = cl(V \cap F) \subset cl(V) \subset X - E$. Then $cl(V) \cap E = \phi$, as desired.

The following result gives a decomposition of meekly π-normality.

Theorem 3.21. A space is π-normal if and only if it is meekly π-normal and quasi normal.
Proof. Let X be a meekly π-normal and quasi normal space. Let A and B be two disjoint closed sets in X in which A is π-closed. By meekly π-normality of X, there exist disjoint open sets U and V such that $cl(U) \cap cl(V) = \phi$, $cl(A \cap U) = A$ and $cl(B \cap V) = B$. Thus $A \subset cl(U)$ and $B \subset cl(V)$. Here cl(U) and cl(V) are disjoint π-closed sets . So by quasi normality, there exist disjoint open sets W_1 and W_2 such that $cl(U) \subset W_1$ and $cl(V) \subset W_2$. Hence X is π-normal.

Corollary 3.22. In a semi-normal and quasi normal space the following statements are equivalent :
(1). X is normal;
(2). X is π-normal;
(3). X is β-normal;
(4). X is meekly π-normal.
Proof. (1) ⇒ (3) ⇒ (4) and (1) ⇒ (2) ⇒ (4) are obvious.
[(4) ⇒ (1)]. Let X be semi-normal, quasi normal and meekly π-normal space. We have to show X is normal. By Theorem 3.21, X is π-normal. Since every π-normal, semi-normal space is normal, so X is normal.
Theorem 3.23. Every semi-normal, meekly π-normal space is α-normal.
Proof. Let X be a semi-normal, meekly π-normal space. Let A and B be two disjoint closed sets in X. Thus $A \subset (X - B)$. By semi-normality, there exists a π-open set F such that $A \subset F \subset (X - B)$. Now A and $(X - F)$ are two disjoint closed sets in X in which $X - F$ is a π-closed set containing B. Thus by meekly π-normality, there exist disjoint open sets U and V such that $cl(U \cap A) = A$, $cl((X - F) \cap V) = X - F$, and $cl(U) \cap cl(V) = \phi$. Here $A = cl(U \cap A) \subset cl(U)$ and $(X - F) = cl((X - F) \cap V) \subset cl(V)$. Thus U and $W = X - cl(U)$ are two disjoint open sets such that $cl(U \cap A) = A$ and $B \subset W$. Therefore, $cl(W \cap B) = B$ and X is α-normal.

Theorem 3.24. Suppose that X and Y are topological spaces, X is meekly π-normal, and $f : X \to Y$ is onto, continuous, open, and closed. Then Y is meekly π-normal.
Proof. Suppose that $E, F \subset Y$ are disjoint closed sets and E is π-closed. Since f is continuous, $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint closed sets. To see that $f^{-1}(E) = cl(f^{-1}(int(E)))$, suppose that $W \subset X$ is open such that $W \cap f^{-1}(E) \neq \phi$. Then f(W) is open in Y and $f(W) \cap E = f(W) \cap cl(int(E)) \neq \phi$ which implies that $f(W) \cap int(E) \neq \phi$. Hence, $W \cap f^{-1}(int(E)) \neq \phi$ and so $f^{-1}(E) = cl(f^{-1}(int(E)))$. Since $f^{-1}(E) = cl(f^{-1}(int(E)))$, $f^{-1}(E)$ is a π-closed set. So there exists an open set $U \subset X$ such that $f^{-1}(F) = cl(f^{-1}(F) \cap U)$ and $cl(U) \cap f^{-1}(E) = \phi$. Since $cl(U) \cap f^{-1}(E) = \phi$, $f(cl(U)) \cap E = \phi$. Also, note that f(U) is open and f(cl(U)) is closed. Since f(cl(U)) is a closed set containing f(U), $cl(f(U)) \subset f(cl(U))$. So $cl(f(U)) \cap E = \phi$. It remains to show that $F = cl(F \cap f(U))$. To see this, let $y \in F$ and O be an open set containing y. Then $f^{-1}(y) \subseteq [f^{-1}(F) \cap f^{-1}(O)]$. Since $f^{-1}(F) = cl(f^{-1}(F) \cap U)$, $f^{-1}(F) \cap U \cap f^{-1}(O) \neq \phi$. Hence, $F \cap f(U) \cap O = f(f^{-1}(F)) \cap f(U) \cap f(f^{-1}(O)) \supset f[f^{-1}(F) \cap U \cap f^{-1}(O)] \neq \phi$, as desired.

REFERENCES

1. *A.V. Arhangel'skii, L. Ludwig , On α-normal and β-normal spaces, Comment. Math. Univ. Carolin. 42:3 (2001) 507–519.*
2. *R.L. Blair, Spaces in which special sets are z-embedded, Canad. J. Math. 28:4 (1976) 673–690.*

3. *N. Bourbaki, Topologie general, 1951, Paris, Actualites Sci. Ind., Nos. Hermann, 858-1142.*
4. *A. Csaszar, General Topology, Adam Higler Ltd, Bristol, 1978.*
5. *A.K. Das, A note on spaces between normal and κ-normal spaces, Filomat 27:1 (2013) 85–88.*
6. *A.K. Das, Pratibha Bhat and J. K. Tartir, On a simultaneous of β-normalily and almost normality, Filomat 31:2 (2017) 425–430.*
7. *J. Dugundji, Topology, Allyn and Bacon, Inc., 470, Atlantic Avenue, Boston, 1966.*
8. *R. Engelking, General Topology, vol. 6, Berlin: Heldermann (Sigma Series in Pure Mathematics), Poland, 1977.*
9. *L. N. Kalantan, π-normal topological spaces, Filomat 22:1 (2008) 173–181.*
10. *J. K. Kohli, A.K. Das, New normality axioms and decompositions of normality, Glasnik Mat. 37(57)(2002) 165–175.*
11. *J. K. Kohli, A.K. Das, On functionally θ-normal spaces, Applied General Topology, 6:1 (2005) 1–14.*
12. *H. Kumar and M. C. Sharma, Softly regular spaces in topological spaces, Journal of Emerging Technologies and Innovative Research, Vol. 6, Issue 11, (2018), 183-190.*
13. *H. Kumar, Epi-π normal spaces in topological spaces, Journal of Emerging Technologies and Innovative Research, Vol. 10, Issue 4, (2023), e656-e669.*
14. *K. Kuratowski, Topologie I, Hafner, New York, 1958.*
15. *E. Murtinova, A β-normal Tyconoff space which is not normal, Comment. Math. Univ. Carolin. 43:1 (2002) 159–164.*
16. *M. K. Singal, S.P. Arya, Almost normal and almost completely regular spaces, Glasnik Mat. 5(25) (1970) 141–152.*
17. *M. K. Singal, A.R. Singal, Mildly normal spaces, Kyungpook Math J. 13 (1973) 27–31.*
18. *M. K. Singal and S. Arya, On almost regular spaces, Glas. Mat. 4 (1969), no. 24, 89–99.*
19. *S. Singh and M. K. Rana, On k-normal spaces, Proyeceiones Journal of Mathematics, Vol. 42, No. 3 (2023), 695-712.*
20. *E.V. Schepin, Real-valued functions, and spaces close to normal, Sib. Matem. Journ. 13:5 (1972) 1182–1196.*
21. *E. V. Schepin, On topological products, groups, and a new class of spaces more general than metric spaces, Soviet Math. Dokl. 17:1 (1976) 152–155.*
22. *M. C. Sharma and Hamant Kumar, Softly normal topological spaces, Acta Ciencia Indica, Vol. XLI, M. No. 2, (2015), 81-84.*
23. *L. A. Steen, J.A. Seebach Jr., Counterexamples in Topology, Springer Verlag, New york, 1978.*
24. *N. V. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl. 78:2 (1968) 103–118.*
25. *V. Zaitsev, On certain classes of topological spaces and their bicompactifications, Dokl. Akad. Nauk SSSR 178 (1968), 778–779.*