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TESTING THE ELONGATION OF 3-TUPLES RELATING TO POLYNOMIALS AND POLYGONAL NUMBERS WITH PECULIAR PROCLAMATION

V. Pandichelvi[1], B. Umamaheswari[2]
[1]*Assistant Professor, PG & Research Department of Mathematics, Urumu Dhanalakshmi College, Trichy.*
(Affiliated to Bharathidasan University)
[2]*Assistant Professor, Department of Mathematics, Meenakshi College of Engineering, Chennai.*

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ABSTRACT
In this paper, special 2-tuples with elements stand well-known polynomials and polygonal numbers with different sides that exclusively meet the property that the summation or difference of two elements provides a square can be stretched to 3-tuples but not 4-tuples with the same characteristics are displayed. Also, the similar characteristics of all such 3-tuples are checked with numerical values by Python programs.
KEYWORDS: *Diophantine* m*-tuples, Congruence relations, polygonal numbers, polynomials*

1. INTRODUCTION

"A set of positive integers $(b_1, b_2, b_3 \ldots, b_m)$ such that $b_i b_j + n$ is a perfect square for all $1 \le i < j \le m$, $n \epsilon Z - \{0\}$ is called a Diophantine m-tuple with property $\mathcal{D}(n)$" see [1]. Many Mathematicians contemplated the existence of Diophantine quadruples with the property $\mathcal{D}(n)$ for any arbitrary integer n and for any polynomial in n [5,9]. Numerous Diophantine triples satisfying various properties are deliberated in [4,7,8]. In [6,10], authors investigated some extendable Diophantine triples satisfying suitable conditions. Various non-extendable Diophantine triples are analysed in [2,3,11].

This paper presents that 2-tuples with members remain renowned polynomials and polygonal numbers together with the property that the totality or difference of two members can be elongated to 3-tuples but not 4-tuples with identical features.

2. RUDIMENTARY DEFINITIONS

2.1. Bernoulli polynomial

The explicit formula for the Bernoulli polynomial is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k$$

and first few Bernoulli polynomials are

$$B_0(x) = 1, B_1(x) = x - \tfrac{1}{2}, B_2(x) = x^2 - x + \tfrac{1}{6} \text{ and } B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x$$

2.2. Euler polynomial

The unambiguous formula for the Euler polynomial is

$$E_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{m-k}$$

and primary Euler polynomials are

$$E_0(x) = 1, E_1(x) = x - \tfrac{1}{2}, E_2(x) = x^2 - x \text{ and } E_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{4}$$

2.3. Chebyshev polynomial

The common formula for the Chebyshev polynomial is

$$U_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-r}{r} (2x)^{n-2r}$$

and principally some Chebyshev polynomials are

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$U_0(x) = 1, U_1(x) = 2x, U_2(x) = 4x^2 - 1$ and $U_3(x) = 8x^3 - 4x$

3. DISCERNMENT OF NON-EXTENDABLE 3-TUPLES

Limited categories of 3-tuples comprising habituated polynomials and special numbers in which the difference and sum of two elements among them provides the square of a polynomial is revealed and it is proved that all these 3-tuples cannot be prolonged into 4-tuples in sections 3.1.1 to 3. 2.2.

3.1.1. Estimation of ephemeral 3-tuples encircling familiar polynomials

Consider $(P_x, Q_x) = \left(6B_2(x), 2E_2(x)\right) = (6x^2 - 6x + 1, 2x^2 - 2x)$ be $2 -$ tuples sustaining the condition that their difference is the square of some polynomial.

Protraction of 2-tuples to 3-tuples

Let R_x be any non-negative integer such that

$$P_x - R_x = \tau_x^2 \quad (1)$$

$$Q_x - R_x = \omega_x^2 \quad (2)$$

Confiscating R_x from (1) and (2) gives that

$$4x^2 - 4x + 1 = \tau_x^2 - \omega_x^2$$

Using the Factorisation method, it is detected that

$$\tau_x = 2x^2 - 2x + 1 \text{ and } \omega_x = 2x^2 - 2x \quad (3)$$

Retentive (3) in (1) offers the possibility of R_x as

$$R_x = -4x^4 + 8x^3 - 2x^2 - 2x \quad (4)$$

Thus, the 3-tuples $(6x^2 - 6x + 1, 2x^2 - 2x, -4x^4 + 8x^3 - 2x^2 - 2x)$ (5)
sustaining the condition that the difference of any two polynomials remains square of some additional polynomial.

Grouping the non-extendibility of 3-tuples

Let S_x be any non-negative integer such that

$$P_x - S_x = \eta_x^2 \quad (6)$$

$$Q_x - S_x = \zeta_x^2 \quad (7)$$

$$R_x - S_x = \delta_x^2 \quad (8)$$

Eradicating S_x from (6) and (8) establishes that

$$4x^4 - 8x^3 + 8x^2 - 4x + 1 = \eta_x^2 - \delta_x^2 \quad (9)$$

Removal of S_x from (7) and (8) provides that

$$4x^4 - 8x^3 + 4x = \zeta_x^2 - \delta_x^2 \quad (10)$$

The conversions of $\zeta_x = \rho + 1$ and $\delta_x = \rho - 1$in (10) yields the option of ρ as cited below

$$\rho = x^4 - 2x^3 + x^2 \quad (11)$$

Thus,

$$\zeta_x = x^4 - 2x^3 + x^2 + 1 \quad \text{and} \quad \delta_x = x^4 - 2x^3 + x^2 - 1$$

In addition, it is evidently perceived that

$$\zeta_x^2 \equiv 1(mod\ 4)$$

$$\delta_x^2 \equiv 1(mod\ 4)$$

Furthermore, it is noticed from (9) that

$$\eta_x^2 = \delta_x^2 + 4x^4 + 8x^3 + 8x^2 + 4x + 1$$

$$\Rightarrow \eta_x^2 \equiv 2(mod\ 4)$$

This is impossible by the rule that any square number is either congruent to 0 or 1 modulo 4.
Thus, η_x is not a square number.
Consequently, it is determined that the 3-tuples exposed in (5) cannot be prolonged into 4-tuples.

3.1.2. Pursuit of 3-tuples encompassing renowned polynomials

Deliberate $(U_x, V_x) = (U_2(x), 2E_2(x))$ be such that $U_x - V_x = (2x - 1)^2$
Applying the procedure as explained in 3.1.1, the 2-tuples (U_x, V_x) can be stretched to 3-tuples $(U_x, V_x, W_x) = (4x^2 - 1, 4x - 2, -4x^4 + 8x^3 - 4x^2 + 4x - 2)$ composed with the condition that the difference between any two polynomials is a perfect square.

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A look at the non-extendable triple

Let Z_x be any non-negative integer such that

$$U_x - Z_x = a_x^2 \quad (12)$$
$$V_x - Z_x = b_x^2 \quad (13)$$
$$W_x - Z_x = c_x^2 \quad (14)$$

Removal of Z_x from above three equations lead to

$$4x^4 - 8x^3 + 8x^2 - 4x + 1 = a_x^2 - c_x^2 \quad (15)$$
$$4x^4 - 8x^3 + 4x^2 = b_x^2 - c_x^2 \quad (16)$$

The modifications of $b_x = X + 1$ and $c_x = X - 1$ in (16) yields

$$b_x = x^4 - 2x^3 + x^2 + 1 \text{ and } c_x = x^4 - 2x^3 + x^2 - 1 \quad (17)$$

Correspondingly, it is attained by

$b_x^2 \equiv 1(mod\ 4)$ and

$c_x^2 \equiv 1(mod\ 4)$

Moreover, it is noted from (15) as

$$a_x^2 = c_x^2 + 4x^4 - 8x^3 + 8x^2 - 4x + 1$$
$$\Rightarrow a_x^2 \equiv 2(mod\ 4)$$

This statement is against the fact that any square number is either congruent to 0 or 1 modulus 4.

Therefore, a_x is not a square inferred that 3-tuples (U_x, V_x, W_x) unable to prolonged as 4-tuples.

3.2.1 Perspicacity of non-extendable 3-tuples concerning distinctive numbers

The $(x+1)^{th}$ term in an arrangement of polygonal numbers with side five demarcated by

$$P_x = \frac{5x^2+5x+2}{2}$$

The x^{th} term in a pattern of Pronic number is described by

$$p_x = x^2 + x$$

Undertake that

$$A_x = 2P_x = 5x^2 + 5x + 2 \quad (18)$$
$$B_x = p_x + 1 = x^2 + x + 1 \quad (19)$$

Select (A_x, B_x) is a couple of numbers composed with the statement that their difference is the square of some polynomial.

In Mathematical notation, it is identified by

$$A_x - B_x = 4x^2 + 4x + 1 = (2x+1)^2 \quad (20)$$

Enlargement of 3-tuples from 2-tuples

Let C_x be any non-negative integer such that

$$A_x - C_x = M_x^2 \quad (21)$$
$$B_x - C_x = N_x^2 \quad (22)$$

Eradication of C_x from (21) and (22) contributes that

$$M_x^2 - N_x^2 = 4x^2 + 4x + 1$$

Using the Factorisation method, it is detected that

$$M_x = 2x^2 + 2x + 1 \text{ and } N_x = 2x^2 + 2x \quad (23)$$

Recollecting the above-professed values of M_x or N_x either in (21) or (22), the couple (A_x, B_x) is protracted into 3-tuples with the following third element

$$C_x = -4x^4 - 8x^3 - 3x^2 + x + 1 \quad (24)$$

Thus, $(5x^2 + 5x + 2, x^2 + x + 1, -4x^4 - 8x^3 - 3x^2 + x + 1)$ is the desired 3-tuples such that the difference between any two members is a leftover square of some polynomial.

Authentication of non-extendibility of the consequent 3-tuples

Let D_x be any non-negative integer such that

$$A_x - D_x = U_x^2 \quad (25)$$
$$B_x - D_x = V_x^2 \quad (26)$$
$$C_x - D_x = W_x^2 \quad (27)$$

Afterward, the execution of elementary arithmetical calculation results

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$$U_x^2 - W_x^2 = 4x^4 + 8x^3 + 8x^2 + 4x + 1 \quad (28)$$
$$V_x^2 - W_x^2 = 4x^4 + 8x^3 + 4x^2 \quad (29)$$

As declared earlier, the factorisation method in (29) yields the values of V_x and W_x as follows

$$V_x = x^4 + 2x^3 + x^2 + 1 \text{ and } W_x = x^4 + 2x^3 + x^2 - 1 \quad (30)$$

Consequently, it is observed from (30) that

$$V_x^2 \equiv 1(mod\ 4) \text{ and } W_x^2 \equiv 1(mod\ 4)$$

Additionally, the replacement of the choice of W_x mentioned in (28), acquires that $U_x^2 \equiv 2(mod\ 4)$
The impossibility of this proclamation concluded that, the succeeding 3-tuples $(5x^2 + 5x + 2, x^2 + x + 1, -4x^4 - 8x^3 - 3x^2 + x + 1)$ satisfying the hypothesis cannot be expanded into 4-tuples.

3.2.2. Clear-sightedness of transient 3-tuples concerning polygonal numbers

The x^{th} terms in a sequence of polygonal numbers with sides ten and six are delineated respectively by

$$D_x = 4x^2 - 3x \text{ and } H_x = 4x + 1$$

Let $(G_x, H_x) = (4D_x, H_x) = (16x^2 - 12x, 4x + 1)$ be a pair of 2-tuples such that their sum is a square of a polynomial namely $(4x - 1)^2$.

Recapping the analogous procedure as illuminated in section 3.1.1, the pair of 2-tuples can be stretched into 3-tuples $(G_x, H_x, J_x) = (16x^2 - 12x, 4x + 1, 64x^4 - 128x^3 + 48x^2 + 12x)$ whereas the addition of two elements among them is a square of a polynomial.

Investigation of the non-extendibility of 3-tuples

Suppose the evaluated 3-tuples can be lengthened into 4-tuples
Let K_x be any non-negative integer such that

$$G_x + K_x = \alpha_x^2 \quad (31)$$
$$H_x + K_x = \beta_x^2 \quad (32)$$
$$J_x + K_x = \gamma_x^2 \quad (33)$$

Amputation of K_x from (31) and (33) delivers that

$$\gamma_x^2 - \alpha_x^2 = 64x^4 - 128x^3 + 32x^2 + 24x \quad (34)$$

Exclusion of K_x from (32) and (33) affords that

$$\gamma_x^2 - \beta_x^2 = 64x^4 - 128x^3 + 48x^2 + 8x - 1 \quad (35)$$

Next implementation of the new transformations $\alpha_x = \Omega - 1$ and $\gamma_x = \Omega + 1$ in (34) propagates that

$$\Omega = 16x^4 - 32x^3 + 8x^2 + 6x \quad (36)$$

Subsequently,

$$\alpha_x = 16x^4 - 32x^3 + 8x^2 + 6x - 1 \text{and } \gamma_x = 16x^4 - 32x^3 + 8x^2 + 6x + 1$$

Also, it is manifestly seen that

$$\alpha_x^2 \equiv 1(mod\ 4) \text{ and } \gamma_x^2 \equiv 1(mod\ 4)$$

Besides, it is perceived from (35) that

$$\beta_x^2 \equiv 2(mod\ 4)$$

This illogicality shows that the succeeding $3 -$ tuples
$(16x^2 - 12x, 4x + 1, 64x^4 - 128x^3 + 48x^2 + 12x)$ can not be extended to $4 -$ tuples.

4. DEMONSTRATION OF PYTHON PROGRAM FOR THE PROPOSALS

```
Import cmath
def is_perfect_square(n):
   root = cmath.sqrt(n)
   return int(root.real) == root.real and int(root.imag) == root.imag
# Get input polynomials
poly1 = input("Enter 1st polynomial: ")
poly2 = input("Enter 2nd polynomial: ")
poly3 = input("Enter 3rd polynomial: ")
```

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```
# Evaluate polynomials
x = 2
p1 = eval(poly1)
p2 = eval(poly2)
p3 = eval(poly3)
# Check differences
count = 0
if not is_perfect_square(p1 - p2):
    print(f"{poly1} - {poly2} is not a perfect square")
    count += 1
if not is_perfect_square(p1 - p3):
    print(f"{poly1} - {poly3} is not a perfect square")
    count += 1
if not is_perfect_square(p2 - p3):
    print(f"{poly2} - {poly3} is not a perfect square")
    count += 1
if count == 0:
    print("The pair of polynomials extends to a triple")
    # Get 4th polynomial
    poly4 = input("Enter 4th polynomial: ")
    p4 = eval(poly4)
    # Check differences
    flag = 0
    if not is_perfect_square(p1 - p4):
        print(f"{poly1} - {poly4} is not a perfect square")
        flag += 1
    if not is_perfect_square(p2 - p4):
        print(f"{poly2} - {poly4} is not a perfect square")
        flag += 1
    if not is_perfect_square(p3 - p4):
        print(f"{poly3} - {poly4} is not a perfect square")
        flag += 1
    print("The pair of polynomials extends to a triple but not to a quadruple")
```

5. CONCLUSION

This study exhibits that special 2-tuples with polygonal numbers and polynomial components that exclusively meet the property that their sum or difference gives a square can be extended to 3-tuples but not 4-tuples with the same feature. To put it all together, one can treasure Diophantine quadruples, Diophantine quintuples, and so on by looking for other polygonal numbers or polynomials that have the right properties.

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