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ON THE NEGATIVE PELL EQUATION $y^2 = 14x^2 - 13$

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ABSTRACT

The binary quadratic equation represented by the Negative Pellian $y^2 = 14x^2 - 13$ ***is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.***

KEYWORDS: ***Binary quadratic, hyperbola, parabola, pell equation, integral solutions. 2010 mathematics subject classification: 11D09***

INTRODUCTION

A binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-15]. In this communication, yet another interesting hyperbola given by $y^2 = 14x^2 - 13$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained.

METHOD OF ANALYSIS

The Negative Pell equation representing hyperbola under consideration is

$$y^2 = 14x^2 - 13 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 1 \, , \, y_0 = 1$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 14x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{14}} g_n \, , \, \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(15 + 4\sqrt{14}\right)^{n+1} + \left(15 - 4\sqrt{14}\right)^{n+1}$$

$$g_n = \left(15 + 4\sqrt{14}\right)^{n+1} - \left(15 - 4\sqrt{14}\right)^{n+1} \; , \; n = -1, 0, 1, 2......$$

Applying Brahmagupta lemma between $\left(x_0, y_0\right)$ and $\left(\tilde{x}_n, \tilde{y}_n\right)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{1}{2\sqrt{14}} g_n$$

$$y_{n+1} = \frac{1}{2} f_n + \frac{\sqrt{14}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 30x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 30y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table: 1 Numerical examples

n	x_{n+1}	y_{n+1}
-1	1	1
0	19	71
1	569	2129
2	17051	63799
3	15311779	57291431

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} and y_{n+1} are always odd.

2. Relations among the solutions

- $x_{n+3} - 30x_{n+2} + x_{n+1} = 0$
- $4y_{n+1} - x_{n+2} + 15x_{n+1} = 0$
- $4y_{n+2} - 15x_{n+2} + x_{n+1} = 0$
- $4y_{n+3} - 449x_{n+2} + 15x_{n+1} = 0$
- $120y_{n+1} - x_{n+3} + 449x_{n+1} = 0$
- $8y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $120y_{n+3} - 449x_{n+3} + 13x_{n+1} = 0$
- $y_{n+2} - 56x_{n+1} - 15y_{n+1} = 0$
- $y_{n+3} - 1680x_{n+1} - 449y_{n+1} = 0$
- $15y_{n+3} - 56x_{n+1} - 449y_{n+2} = 0$
- $449x_{n+3} - x_{n+1} + 120y_{n+3} = 0$
- $4y_{n+1} - 15x_{n+3} + 449x_{n+2} = 0$
- $4y_{n+2} - x_{n+3} + 15x_{n+2} = 0$
- $4y_{n+3} - 15x_{n+3} + x_{n+2} = 0$
- $15y_{n+2} - 56x_{n+2} - y_{n+1} = 0$
- $y_{n+3} - 112x_{n+2} - y_{n+1} = 0$
- $y_{n+3} - 56x_{n+2} - 15y_{n+2} = 0$
- $449y_{n+2} - 56x_{n+3} - 15y_{n+1} = 0$
- $449y_{n+3} - 1680x_{n+3} - y_{n+1} = 0$

- $15y_{n+3} - 56x_{n+3} - y_{n+2} = 0$
- $y_{n+3} - 30y_{n+2} + y_{n+1} = 0$

3. Each of the following expressions represents a nasty number

- $\frac{3}{13}\left(71x_{2n+2} - x_{2n+3} + 52\right)$
- $\frac{1}{130}\left(2129x_{2n+2} - x_{2n+4} + 1560\right)$
- $\frac{6}{13}\left(28x_{2n+2} - 2y_{2n+2} + 26\right)$
- $\frac{2}{65}\left(532x_{2n+2} - 2y_{2n+3} + 390\right)$
- $\frac{6}{5837}\left(15932x_{2n+2} - 2y_{2n+4} + 11674\right)$
- $\frac{3}{26}\left(4258x_{2n+3} - 142x_{2n+4} + 104\right)$
- $\frac{2}{65}\left(28x_{2n+3} - 142y_{2n+2} + 390\right)$
- $\frac{6}{13}\left(532x_{2n+3} - 142y_{2n+3} + 26\right)$
- $\frac{2}{65}\left(15932x_{2n+3} - 142y_{2n+4} + 390\right)$
- $\frac{6}{5837}\left(28x_{2n+4} - 4258y_{2n+2} + 11674\right)$
- $\frac{2}{65}\left(532x_{2n+4} - 4258y_{2n+3} + 390\right)$
- $\frac{6}{13}\left(15932x_{2n+4} - 4258y_{2n+4} + 26\right)$
- $\frac{3}{26}\left(2y_{2n+3} - 38y_{2n+2} + 104\right)$

- $\frac{1}{260}\left(2y_{2n+4}-1138y_{2n+2}+3120\right)$
- $\frac{3}{26}\left(38y_{2n+4}-1138y_{2n+3}+104\right)$

4. Each of the following expressions represents a cubical integer

- $\frac{1}{26}\left(71x_{3n+3}-x_{3n+4}+213x_{n+1}-3x_{n+2}\right)$
- $\frac{1}{52}\left[4258x_{3n+4}-142x_{3n+5}+12774x_{n+2}-426x_{n+3}\right]$
- $\frac{1}{780}\left(2129x_{3n+3}-x_{3n+5}+6387x_{n+1}-3x_{n+3}\right)$
- $\frac{1}{13}\left[28x_{3n+3}-2y_{3n+3}+84x_{n+1}-6y_{n+1}\right]$
- $\frac{1}{195}\left[532x_{3n+3}-2y_{3n+4}+1596x_{n+1}-6y_{n+2}\right]$
- $\frac{1}{5837}\left(15932x_{3n+3}-2y_{3n+5}+47796x_{n+1}-6y_{n+3}\right)$
- $\frac{1}{195}\left[28x_{3n+4}-142y_{3n+3}+84x_{n+2}-426y_{n+1}\right]$
- $\frac{1}{13}\left(532x_{3n+4}-142y_{3n+4}+1596x_{n+2}-426y_{n+2}\right)$
- $\frac{1}{195}\left[15932x_{3n+4}-142y_{3n+5}+47796x_{n+2}-426y_{n+3}\right]$
- $\frac{1}{5837}\left[28x_{3n+5}-4258y_{3n+3}+84x_{n+3}-12774y_{n+1}\right]$
- $\frac{1}{195}\left[532x_{3n+5}-4258y_{3n+4}+1596x_{n+3}-12774y_{n+2}\right]$
- $\frac{1}{13}\left(15932x_{3n+5}-4258y_{3n+5}+47796x_{n+3}-12774y_{n+3}\right)$

- $\frac{1}{52}\left[2y_{3n+4}-38y_{3n+3}+6y_{n+2}-114y_{n+1}\right]$
- $\frac{1}{1560}\left[2y_{3n+5}-1138y_{3n+3}+6y_{n+3}-3414y_{n+1}\right]$
- $\frac{1}{52}\left[38y_{3n+5}-1138y_{3n+4}+114y_{n+3}-3414y_{n+2}\right]$

5. Each of the following expressions represents a bi-quadratic integer

- $\frac{1}{5837}\left(28x_{4n+6}-4258y_{4n+4}+112x_{2n+4}-17032y_{2n+2}+35022\right)$
- $\frac{1}{26}\left[71x_{4n+4}-x_{4n+5}+284x_{2n+2}+4x_{2n+3}+156\right]$
- $\frac{1}{780}\left(2129x_{4n+4}-x_{4n+6}+8516x_{2n+2}-4x_{2n+4}+4680\right)$
- $\frac{1}{13}\left[28x_{4n+4}-2y_{4n+4}+112x_{2n+2}-8y_{2n+2}+78\right]$
- $\frac{1}{195}\left[532x_{4n+4}-2y_{4n+5}+2128x_{2n+2}-8y_{2n+3}+1170\right]$
- $\frac{1}{52}\left(4258x_{4n+5}-142x_{4n+6}+17032x_{2n+3}-568x_{2n+4}+312\right)$
- $\frac{1}{195}\left[28x_{4n+5}-142y_{4n+4}+112x_{2n+3}-568y_{2n+2}+1170\right]$
- $\frac{1}{13}\left(532x_{4n+5}-142y_{4n+5}+2128x_{2n+3}-568y_{2n+3}+78\right)$
- $\frac{1}{195}\left[15932x_{4n+5}-142y_{4n+6}+63728x_{2n+3}-568y_{2n+4}+1170\right]$
- $\frac{1}{195}\left[532x_{4n+6}-4258y_{4n+5}+2128x_{2n+4}-17032y_{2n+3}+1170\right]$
- $\frac{1}{13}\left[15932x_{4n+6}-4258y_{4n+6}+63728x_{2n+4}-17032y_{2n+4}+78\right]$
- $\frac{1}{52}\left(2y_{4n+5}-38y_{4n+4}+8y_{2n+3}-152y_{2n+2}+312\right)$
- $\frac{1}{1560}\left[2y_{4n+6}-1138y_{4n+4}+8y_{2n+4}-4552y_{2n+2}+9360\right]$

- $\frac{1}{52}\left[38y_{4n+6}-1138y_{4n+5}+152y_{2n+4}-4552y_{2n+3}+312\right]$
- $\frac{1}{7}\left[155y_{4n+5}-13y_{4n+6}+620y_{2n+3}-52y_{2n+4}+42\right]$
- $\frac{1}{5837}\left(15932x_{4n+4}-2y_{4n+6}+63728x_{2n+2}-8y_{2n+4}+35022\right)$

6. Each of the following expressions represents a quintic integer

- $\frac{1}{52}\left(38y_{5n+7}-1138y_{5n+6}+190y_{3n+5}-5690y_{3n+4}+380y_{n+3}-11380y_{n+2}\right)$
- $\frac{1}{26}\left[71x_{5n+5}-x_{5n+6}+355x_{3n+3}-5x_{3n+4}+710x_{n+1}-10x_{n+2}\right]$
- $\frac{1}{780}\left(2129x_{5n+5}-x_{5n+7}+10645x_{3n+3}-5x_{3n+5}+21290x_{n+1}-10x_{n+3}\right)$
- $\frac{1}{13}\left[28x_{5n+5}-2y_{5n+5}+140x_{3n+3}-10y_{3n+3}+280x_{n+1}-20y_{n+1}\right]$
- $\frac{1}{195}\left[532x_{5n+5}-2y_{5n+6}+2660x_{3n+3}-10y_{3n+4}+5320x_{n+1}-20y_{n+2}\right]$
- $\frac{1}{52}\left(4258x_{5n+6}-142x_{5n+7}+21290x_{3n+4}-710x_{3n+5}+42580x_{n+2}-1420x_{n+3}\right)$
- $\frac{1}{195}\left[28x_{5n+6}-142y_{5n+5}+140x_{3n+4}-710y_{3n+3}+280x_{n+2}-140y_{n+1}\right]$
- $\frac{1}{13}\left(532x_{5n+6}-142y_{5n+6}+2660x_{3n+4}-710y_{3n+4}+5320x_{n+2}-1420y_{n+2}\right)$
- $\frac{1}{195}\left[15932y_{5n+6}-142y_{5n+7}+79660x_{3n+4}-710y_{3n+5}+159320x_{n+2}-1420y_{n+3}\right]$
- $\frac{1}{5837}\left[28x_{5n+7}-4258y_{5n+5}+140x_{3n+5}-21290y_{3n+3}+280x_{n+3}-42580y_{n+1}\right]$
- $\frac{1}{195}\left[532x_{5n+7}-4258y_{5n+6}+2660x_{3n+5}-21290y_{3n+4}+5320x_{n+3}-42580y_{n+2}\right]$
- $\frac{1}{1560}\left[2y_{5n+7}-1138y_{5n+5}+10y_{3n+5}-5690y_{3n+3}+20y_{n+3}-11380y_{n+1}\right]$

- $\frac{1}{13}\left(15932x_{5n+7}-4258y_{5n+7}+79660x_{3n+5}-21290y_{3n+5}+159320x_{n+3}-42580y_{n+3}\right)$

- $\frac{1}{5837}\left(15932x_{5n+5}-2y_{5n+7}+79660x_{3n+3}-10y_{3n+5}+159320x_{n+1}-20y_{n+3}\right)$

REMARKABLE OBSERVATIONS

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. No	Hyperbolas	(X, Y)
1	$X^2-14Y^2=2704$	$(71x_{n+1}-x_{n+2}, x_{n+2}-19x_{n+1})$
2	$X^2-14Y^2=2433600$	$(2129x_{n+1}-x_{n+3}, x_{n+3}-569x_{n+1})$
3	$X^2-14Y^2=676$	$(28x_{n+1}-2y_{n+1}, 2y_{n+1}-2x_{n+1})$
4	$X^2-14Y^2=152100$	$(532x_{n+1}-2y_{n+2}, 2y_{n+2}-142x_{n+2})$
5	$X^2-14Y^2=136282276$	$(15932x_{n+1}-2y_{n+3}, 2y_{n+3}-4258x_{n+1})$
6	$X^2-14Y^2=10816$	$(4258x_{n+2}-142x_{n+3}, 38x_{n+3}-1138x_{n+2})$
7	$X^2-14Y^2=152100$	$(28x_{n+2}-142y_{n+1}, 38y_{n+1}-2x_{n+2})$
8	$X^2-14Y^2=676$	$(532x_{n+2}-142x_{n+2}, 38y_{n+2}-142x_{n+2})$
9	$X^2-14Y^2=152100$	$(15932x_{n+3}-142y_{n+3}, 38y_{n+3}-4258x_{n+2})$
10	$X^2-14Y^2=136282276$	$(28x_{n+3}-4258y_{n+1}, 1138y_{n+1}-2x_{n+3})$
11	$X^2-14Y^2=152100$	$(532x_{n+3}-4258y_{n+2}, 1138y_{n+2}-142x_{n+3})$
12	$X^2-14Y^2=676$	$(15932x_{n+3}-4258y_{n+3}, 1138y_{n+3}-4258x_{n+3})$
13	$14X^2-Y^2=151424$	$(2y_{n+2}-38y_{n+1}, 142y_{n+1}-2y_{n+2})$
14	$14X^2-Y^2=136281600$	$(2y_{n+3}-1138y_{n+1}, 4258y_{n+1}-2y_{n+3})$
15	$14X^2-Y^2=151424$	$(28y_{n+3}-1138y_{n+2}, 4258y_{n+2}-142y_{n+3})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabolas	(X, Y)
1	$26X - 14Y^2 = 1352$	$(71x_{2n+2} - x_{2n+3}, x_{n+2} - 19x_{n+1})$
2	$780X - 14Y^2 = 1216800$	$(2129x_{2n+2} - x_{2n+4}, x_{n+3} - 569x_{n+1})$
3	$13X - 14Y^2 = 338$	$(28x_{2n+2} - 2y_{2n+2}, 2y_{n+1} - 2x_{n+1})$
4	$195X - 14Y^2 = 76050$	$(532x_{2n+2} - 2y_{2n+3}, 2y_{n+2} - 142x_{n+2})$
5	$5837X - 14Y^2 = 68141138$	$(15932x_{2n+2} - 2y_{2n+4}, 2y_{n+3} - 4258x_{n+1})$
6	$52X - 14Y^2 = 5408$	$(4258x_{2n+3} - 142x_{2n+4}, 38x_{n+3} - 1138x_{n+2})$
7	$195X - 14Y^2 = 76050$	$(28x_{2n+3} - 142y_{2n+2}, 38y_{n+1} - 2x_{n+2})$
8	$13X - 14Y^2 = 338$	$(532x_{2n+3} - 142x_{2n+3}, 38y_{n+2} - 142x_{n+2})$
9	$195X - 14Y^2 = 76050$	$(15932x_{2n+4} - 142y_{2n+4}, 38y_{n+3} - 4258x_{n+2})$
10	$5837X - 14Y^2 = 68141138$	$(28x_{2n+4} - 4258y_{2n+2}, 1138y_{n+1} - 2x_{n+3})$
11	$195X - 14Y^2 = 76050$	$(532x_{2n+4} - 4258y_{2n+3}, 1138y_{n+2} - 142x_{n+3})$
12	$13X - 14Y^2 = 338$	$(15932x_{2n+4} - 4258y_{2n+4}, 1138y_{n+3} - 4258x_{n+3})$
13	$728X - Y^2 = 75712$	$(2y_{2n+3} - 38y_{2n+2}, 142y_{n+1} - 2y_{n+2})$
14	$21840X - Y^2 = 68140800$	$(2y_{2n+4} - 1138y_{2n+2}, 4258y_{n+1} - 2y_{n+3})$
15	$728X - Y^2 = 75712$	$(28y_{2n+4} - 1138y_{2n+3}, 4258y_{n+2} - 142y_{n+3})$

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