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ON THE TERNARY SURD EQUATION

$$x+\sqrt{x}+2(y+\sqrt{y})=z+\sqrt{z}$$

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ABSTRACT

The transcendental equation with three unknowns given by $x+\sqrt{x}+2(y+\sqrt{y})=z+\sqrt{z}$ *is studied for obtaining its non-zero integer solutions.*

INTRODUCTION

The subject of Diophantine equation, one of the interesting areas of Number Theory, plays a significant role in higher arithmetic and has a marvellous effect on credulous people and always occupies a remarkable position due to unquestioned historical importance. The Diophantine equations may be either polynomial equation with at least two unknowns for which integer solution, are required or transcendental equation involving trigonometric, logarithmic, exponential and surd function such that one may be interested in getting integer solution. Most of the Diophantine problems solved by the researchers are polynomial equations [3-7]. Note that, the transcendental equation can be solved by transforming it into an equivalent polynomial equation. Adhoc methods exists for some classes of transcendental equations in one variable to transform them into polynomial equations which then might be solved. Some transcendental equation in more than one unknown can be solved by separation of the unknowns reducing them to polynomial equations. In this context, one may refer [1,2,8-15].

In this paper, we are interested in obtaining integer solutions to transcendental equation involving surds. In particular, we obtain different sets of integer solutions to the transcendental equation with three unknowns given by $x+\sqrt{x}+2(y+\sqrt{y})=z+\sqrt{z}$.

Method of analysis

The surd equation with three unknowns under consideration is

$$x+\sqrt{x}+2(y+\sqrt{y})=z+\sqrt{z} \tag{1}$$

The substitution

$$x=u^2, y=v^2, z=w^2 \tag{2}$$

in (1) leads to the ternary quadratic equation

$$(2u+1)^2+2(2v+1)^2=(2w+1)^2+2 \tag{3}$$

Two different methods of solving (3) are illustrated below:

Method 1

Let

$$2u+1=U,\ 2v+1=V,\ 2w+1=P. \tag{4}$$

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On completing the squares, (3) is written as

$$U^2 + 2V^2 = (P^2 + 2)*1 \tag{5}$$

Assume

$$1 = \frac{\left(2r^2 - s^2 + i2rs\sqrt{2}\right)\left(2r^2 - s^2 - i2rs\sqrt{2}\right)}{\left(2r^2 + s^2\right)^2} \tag{6}$$

Substituting (6) in (5) and applying factorization, consider

$$U + i\sqrt{2}V = (P + i\sqrt{2})*\frac{(2r^2 - s^2 + i2\sqrt{2}rs)}{(2r^2 + s^2)}$$

Equating the coefficients of corresponding terms and using (4) for P, we have

$$\begin{aligned} U &= \frac{1}{(2r^2 + s^2)}[2(2r^2 - s^2)w + 2r^2 - s^2 - 4rs] \\ V &= \frac{1}{(2r^2 + s^2)}[2r^2 - s^2 + 2rs + 4rsw] \end{aligned} \tag{7}$$

It is possible to choose the values for r, s, w, such that U, V are integers. In view of (4) and (2), the corresponding integer solutions to (1) are given by

r	s	w	u	v	x	y	z
1	1	3s+3	s	2s+2	s^2	$(2s+2)^2$	$(3s+3)^2$
2	3	17s+13	-s-2	12s+9	$(-s-2)^2$	$(12s+9)^2$	$(17s+13)^2$

Method 2

Taking

$$U = 2u + 1\,, V = 2v + 1\,, P = 2w + 1 \tag{8}$$

in (3), we have

$$U^2 + 2V^2 = P^2 + 2 \tag{9}$$

Choosing

$$P = k\,V\,, k > 1 \tag{10}$$

in (9), we get

$$U^2 = (k^2 - 2)\,V^2 + 2 \tag{11}$$

which is satisfied by

$$V_0 = 1,\ U_0 = k$$

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To obtain the other solutions, consider the Pell equation,

$$U^2 = (k^2 - 2)\, V^2 + 1$$

whose initial solutions are

$$\tilde{V}_0 = k,\ \tilde{U}_0 = k^2 - 1$$

and, the general solution (U_n, V_n) is given by

$$U_{n+1} = \frac{k f_n}{2} + \frac{\sqrt{k^2-2}}{2} g_n,\ V_{n+1} = \frac{1}{2} f_n + \frac{k g_n}{2\sqrt{k^2-2}} \tag{12}$$

where

$$f_n = (k^2 - 1 + k\sqrt{k^2-1})^{n+1} + (k^2 - 1 - k\sqrt{k^2-1})^{n+1},$$
$$g_n = (k^2 - 1 + k\sqrt{k^2-1})^{n+1} - (k^2 - 1 - k\sqrt{k^2-1})^{n+1}$$

In view of (8), it is seen that

$$u_{n+1} = \frac{\left(k f_n + \left(\sqrt{k^2-2}\right) g_n - 2\right)}{4},$$

$$v_{n+1} = \frac{\left(f_n + \frac{k}{\left(\sqrt{k^2-2}\right)} g_n - 2\right)}{4},$$

$$w_{n+1} = \frac{\left(k f_n + \frac{k^2}{\left(\sqrt{k^2-2}\right)} g_n - 2\right)}{4},\ n = 0,1,2,...$$

Using the above values in (2), the integer solutions are obtained when k is odd.

CONCLUSION

In this paper, the transcendental equation involving surds with three unknowns given by $x + \sqrt{x} + 2(y + \sqrt{y}) = z + \sqrt{z}$ has been reduced to polynomial equations of degree two with three unknowns as well as pellian equation for which integer solutions are obtained in an elegant way through suitable transformations.

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