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ON THE SYSTEM OF DOUBLE EQUATIONS $x+y=z+w, y+z=(x+w)^2$

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ABSTRACT
In this paper, different methods to obtain non-zero distinct integer solutions to the system of double equations $x+y=z+w, y+z=(x+w)^2$ are illustrated.

KEYWORDS: ***System of double equations, integer solutions.***

INTRODUCTION

Systems of indeterminate quadratic equations of the form $ax+c=u^2$, $bx+d=v^2$ where a, b, c, d are non-zero distinct constants, have been investigated for solutions by several authors [1, 2] and with a few possible exceptions, most of them were primarily concerned with rational solutions. Even those existing works wherein integral solutions have been attempted, deal essentially with specific cases only and do not exhibit methods of finding integral solutions is a general form. In [3], a general form of the integral solutions to the system of equations $ax+c=u^2$, $bx+d=v^2$ where a, b, c, d are non-zero distinct constants is presented when the product ab is a square free integer whereas the product cd may or may not a square integer. For other forms of system of double diophantine equations, one may refer [4-25].

This communication concerns with yet another interesting system of double Diophantine equations namely $x+y=z+w, y+z=(x+w)^2$ for its infinitely many non-zero distinct integer solutions.

METHOD OF ANALYSIS

Let x, y, z and w be four non-zero distinct integers such that the equations

$$x+y=z+w \quad (1)$$

$$y+z=(x+w)^2 \quad (2)$$

are satisfied. Different methods to obtain non-zero distinct integer values to x, y, z and w satisfying (1) and (2) are exhibited below:

Method 1:

Eliminating y between (1) and (2), the resulting equation is

$$x^2 + (2w+1)x + (w^2 - 2z - w) = 0 \tag{3}$$

Treating (3) as a quadratic in x and solving for x, one obtains

$$x = \frac{1}{2}\left[(-2w-1) \pm \sqrt{8z+8w+1}\,\right] \tag{4}$$

The square-root on the R.H.S of (4) is eliminated when

$$z = m, w = \frac{1}{2}\left(n^2 + 3n - 2m + 2\right) \tag{5}$$

From (4) and (5), we get

$$x = \frac{1}{2}\left(-n^2 - n + 2m\right), -\frac{1}{2}\left(n^2 + 5n - 2m + 6\right) \tag{6}$$

In view of (1), note that

$$y = n^2 + 2n - m + 1,\ n^2 + 4n - m + 4 \tag{7}$$

Thus, (5) , (6) and (7) give two sets of non-zero distinct integer solutions to the system of equations (1) and (2).

Method 2:

The introduction of the transformations

$$x = u + v, w = u - v, z = 4k, y = 4l, (u \neq v \neq 0), (k \neq l \neq 0) \tag{8}$$

in (1) and (2) leads respectively to the equations

$$v = 2(k - l) \tag{9}$$

and

$$u^2 = k + l \tag{10}$$

Observe that (10) is satisfied when

$$l = m, k = (n+1)^2 - m, u = (n+1) \tag{11}$$

and from (9), we have

$$v = 2\left[(n+1)^2 - 2m\right] \tag{12}$$

Using (11) and (12) in (8), we get

$$x = 2n^2 + 5n - 4m + 3$$

$$y = 4m$$

$$z = 4n^2 + 8n - 4m + 4$$

$$w = -2n^2 - 3n + 4m - 1$$

which satisfy (1) and (2).

Method 3:

Consider the transformations

$$x = p + q, y = p - q, z = p + s, w = p - s, (p \neq q \neq s \neq 0) \tag{13}$$

it is seen that (1) is automatically satisfied.

The substitution of (13) in (2) leads to

$$4p^2 + p[4(q-s) - 2] + (q-s)^2 + (q-s) = 0 \tag{14}$$

which is a quadratic in p and solving for p, we get,

$$p = \frac{1}{4}\left\{[2(s-q)+1] \pm \sqrt{1 - 8q + 8s}\,\right\} \tag{15}$$

The square-root on the R.H.S of (15) is eliminated when

$$q = m, s = \frac{1}{2}\left(n^2 - n + 2m\right) \tag{16}$$

From (15) and (16) we have,

$$p = \frac{1}{4}\left(n^2 + n\right), \frac{1}{4}\left(n^2 - 3n + 2\right) \tag{17}$$

Substituting (16) and (17) in (13), there are two sets of solutions to (1) and (2) and they are represented as below:

Set 1:

$$x = \frac{1}{4}\left(n^2 + n\right) + m$$

$$y = \frac{1}{4}\left(n^2 + n\right) - m$$

$$z = \frac{1}{4}\left(3n^2 - n\right) + m$$

$$w = \frac{1}{4}\left(-n^2 + 3n\right) - m$$

where $n, m \neq 0$

Note that, for the values of x, y, z and w to be in integers, choose n such that

$n \equiv 0, -1 \pmod 4$ and $m \in z - \{0\}$

Set 2:

$$x = \frac{1}{4}\left[(n-1)(n-2)\right] + m$$

$$y = \frac{1}{4}\left[(n-1)(n-2)\right] - m$$

$$z = \frac{1}{4}\left(3n^2 - 5n + 2\right) + m$$

$$w = \frac{1}{4}\left(-n^2 - n + 2\right) - m$$

where $n, m \neq 0$

In this case for integer solutions n should be such that

$n \equiv 1, 2 \pmod 4$ and $m \in z - \{0\}$.

However, by treating (14) as a quadratic in q, s in turn and following the above procedure different sets of values of x, y, z and w satisfying (1) and (2) are exhibited below in Table 1:

Table 1: Solutions

Set	x	y	z	w
3	$-4n^2+5n+s-1$	$12n^2-7n-s+1$	$4n^2-n+s$	$4n^2-n-s$
4	$-4n^2-3n+s$	$12n^2+n-s$	$4n^2-n+s$	$4n^2-n-s$
5	$-4n^2+3n+s$	$12n^2-n-s$	$4n^2+n+s$	$4n^2+n-s$
6	$-4n^2-5n+s-1$	$12n^2+7n-s+1$	$4n^2+n+s$	$4n^2+n-s$
7	$4n^2-n+q$	$4n^2-n-q$	$12n^2+n+q$	$-4n^2-3n-q$
8	$4n^2-n+q$	$4n^2-n-q$	$12n^2-7n+q+1$	$-4n^2+5n-q-1$
9	$4n^2+n+q$	$4n^2+n-q$	$12n^2+7n+q+1$	$-4n^2-5n-q-1$
10	$4n^2+n+q$	$4n^2+n-q$	$12n^2-n+q$	$-4n^2+3n-q$

CONCLUSION

In this paper an attempt has been made to obtain all possible integer values of x, y, z and w satisfying (1) and (2). In conclusion one may search for other choices of integer solutions to the system of equations under consideration.

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