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A SEARCH FOR INTEGER SOLUTIONS TO TERNARY BI-QUADRATIC EQUATION

$$(a+1)(x^2+y^2)-(2a+1)\,xy=\left[p^2+(4a+3)q^2\right]z^4$$

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ABSTRACT

This paper aims at determining non-zero distinct integer solutions to the algebraic equation of degree four with three unknowns given by

$$(a+1)(x^2+y^2)-(2a+1)xy=\left[p^2+(4a+3)q^2\right]z^4,a\geq 0$$

KEY WORDS: ***Bi-quadratic with three unknowns, integer solutions, non-homogeneous bi-quadratic.***

1. INTRODUCTION

The subject of diophantine equations in Number Theory is the study of integer solutions to equations. The study of diophantine equations have spanned centuries and is one of the area of Number Theory that has attracted many mathematicians. It is quite obvious and well-known that biquadratic equations are rich in variety. In [1,9], integer solutions to some bi-quadratic equations with three unknowns are presented. In, this paper, the bi-quadratic equation with three unknowns given by $(a+1)(x^2+y^2)-(2a+1)xy=\left[p^2+(4a+3)q^2\right]z^4$ is studied for its distinct integer solutions.

2. METHOD OF ANALYSIS

The fourth degree equation with three unknowns to be solved is

$$(a+1)(x^2+y^2)-(2a+1)xy=\left[p^2+(4a+3)q^2\right]z^4 \tag{1}$$

Different sets of integer solutions to (1) are illustrated below:

Set 1:

The choice

$$x=u+v\,,\ y=u-v,\ u\neq v\neq 0 \tag{2}$$

in (1) leads to

$$u^2+(4a+3)v^2=\left[p^2+(4a+3)q^2\right]z^4 \tag{3}$$

Take

$$z=\alpha^2+(4a+3)\beta^2 \tag{4}$$

Substituting (4) in (3) and factorizing, the resulting equation is written as the system of double equations

$$\left(u+i\sqrt{4a+3}\,v\right)=\left(p+i\sqrt{4a+3}\,q\right)\left(\alpha+i\sqrt{4a+3}\,\beta\right)^4 \tag{5}$$

$$\left(u-i\sqrt{4a+3}\,v\right)=\left(p-i\sqrt{4a+3}\,q\right)\left(\alpha-i\sqrt{4a+3}\,\beta\right)^4 \tag{6}$$

On equating the rational and irrational parts either in (5) or (6), we have

$$\left.\begin{aligned} u &= p\alpha^4-6p(4a+3)\alpha^2\beta^2+p(4a+3)^2\beta^4-4(4a+3)\alpha^3\beta q \\ &\quad +4(4a+3)^2\alpha\beta^3 q \\ v &= 4p\alpha^3\beta-4p(4a+3)\alpha\beta^3+\alpha^4 q-6(4a+3)\alpha^2\beta^2 q \\ &\quad +(4a+3)^2 q\beta^4 \end{aligned}\right\} \tag{7}$$

From (7) and (2), we get

$$\left.\begin{aligned} x &= \left\{\begin{aligned} &\alpha^4(p+q)-6(4a+3)(p+q)\alpha^2\beta^2+(4a+3)^2(p+q)\beta^4 \\ &+4[p-(4a+3)q]\alpha^3\beta+4\left[(4a+3)^2q-(4a+3)p\right]\alpha\beta^3 \end{aligned}\right\} \\ y &= \left\{\begin{aligned} &(p-q)\alpha^4+6(4a+3)(q-p)\alpha^2\beta^2+(4a+3)^2(p-q)\beta^4 \\ &-4[p+(4a+3)q]\alpha^3\beta+4\left[(4a+3)^2q+(4a+3)p\right]\alpha\beta^3 \end{aligned}\right\} \end{aligned}\right\} \tag{8}$$

Thus, (4) and (8) represents the integer solutions to (1).

Set 2:

Observe that (3) is written in the form of ratio as

$$\frac{u+pz^2}{qz^2+v}=\frac{(4a+3)(qz^2-v)}{u-pz^2}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the system of double equations

$$\beta u-\alpha v+(p\beta-\alpha q)z^2=0$$
$$u\alpha+(4a+3)\beta v-[p\alpha+(4a+3)q\beta]z^2=0$$

Applying the method of cross multiplication we have

$$\left.\begin{aligned} u&=p\alpha^2-(4a+3)p\beta^2+2(4a+3)\alpha\beta q\\ v&=-\alpha^2 q+(4a+3)q\beta^2+2\alpha\beta p\end{aligned}\right\} \quad (9)$$

$$z^2=\alpha^2+(4a+3)\beta^2 \quad (10)$$

Note that (10) is satisfied by

$$\beta=2mn,\alpha=(4a+3)m^2-n^2 \quad (11)$$

$$z=(4a+3)m^2+n^2 \quad (12)$$

Substituting the values of α and β from (11) in (9) we get,

$$\left.\begin{aligned} u&=p\left[(4a+3)^2m^4+n^4-2(4a+3)m^2n^2\right]-4(4a+3)pm^2n^2\\ &\quad+4(4a+3)\left[(4a+3)m^3n-mn^3\right]q\\ v&=-q\left[(4a+3)^2m^4+n^4-2(4a+3)m^2n^2\right]+4(4a+3)qm^2n^2\\ &\quad+4\left[(4a+3)m^3n-mn^3\right]p\end{aligned}\right\} \quad (13)$$

Substitution of (13) in (2) gives

$$\left.\begin{aligned} x&=(p-q)\left[(4a+3)^2m^4+n^4-2(4a+3)m^2n^2\right]+4(4a+3)(q-p)m^2n^2\\ &\quad+4\left[(4a+3)m^3n-mn^3\right][P+(4a+3q)]\\ y&=(p+q)\left[(4a+3)^2m^4+n^4-2(4a+3)m^2n^2\right]-4(4a+3)(p+q)m^2n^2\\ &\quad+4\left[(4a+3)m^3n-mn^3\right][(4a+3)q-p]\end{aligned}\right\} \quad (14)$$

Thus, (12) and (14) represent the integer solutions to (1).

Note 1:

Also, (3) is written in the form of ratio as

$$\frac{u+pz^2}{(4a+3)(qz^2+v)}=\frac{qz^2-v}{u-pz^2}=\frac{\alpha}{\beta},\beta\neq 0$$

In this case, the corresponding integer solutions to (1) are given by

$$x = 6(4a+3)(p-q)m^2n^2 + (4a+3)^2(q-p)m^4 + (q-p)n^4 + 4(4a+3)[p+(4a+3)q]m^3n$$
$$-4[p+(4a+3)q]mn^3$$
$$y = 6(4a+3)(p+q)m^2n^2 - (4a+3)^2(p+q)m^4 - (p+q)n^4 + 4(4a+3)[(4a+3)q-p]m^3n$$
$$+4[p-(4a+3)q]mn^3$$
$$z = \beta^2 + (4a+3)\alpha^2$$

Set 3:

write (3) as

$$u^2 + (4a+3)v^2 = \left[p^2 + (4a+3)q^2\right] z^4 * 1 \tag{15}$$

Assume

$$1 = \frac{\left[(2a+1)+i\sqrt{4a+3}\right]\left[(2a+1)-i\sqrt{4a+3}\right]}{(2a+2)^2} \tag{16}$$

Substituting (4) and (16) in (15) and employing the method of factorization, define

$$\left(u+i\sqrt{4a+3}v\right)\left(u-i\sqrt{4a+3}v\right) = \left(p+i\sqrt{4a+3}q\right)\left(p-i\sqrt{4a+3}q\right)\left(\alpha+i\sqrt{4a+3}\beta\right)^4$$
$$\left(\alpha+i\sqrt{4a+3}\beta\right)^4 * \frac{\left[(2a+1)+i\sqrt{4a+3}\right]\left[(2a+1)-i\sqrt{4a+3}\right]}{(2a+2)^2}$$

Equating the positive and negative terms in the above equation, we get

$$\left(u+i\sqrt{4a+3}\,v\right) = \frac{1}{(2a+2)}\left\{\begin{array}{l}\left(p+i\sqrt{4a+3}q\right)\left(\alpha+i\sqrt{4a+3}\beta\right)^4 \\ \left[(2a+1)+i\sqrt{4a+3}\right]\end{array}\right\} \tag{17}$$

$$\left(u-i\sqrt{4a+3}\,v\right) = \frac{1}{(2a+2)}\left\{\begin{array}{l}\left(p-i\sqrt{4a+3}q\right)\left(\alpha-i\sqrt{4a+3}\beta\right)^4 \\ \left[(2a+1)-i\sqrt{4a+3}\right]\end{array}\right\} \tag{18}$$

Equating the real and imaginary parts in (17) or (18), we have

$$\left.\begin{array}{l} u = \dfrac{1}{(2a+2)}\left\{\begin{array}{l}[(2a+1)p-(4a+3)q]\alpha^4 - 6(4a+3)[(2a+1)p-(4a+3)q]\alpha^2\beta^2 \\ +(4a+3)^2[(2a+1)p-(4a+3)q]\beta^4 - 4(4a+3)[(2a+1)q+p]\alpha^3\beta \\ +4(4a+3)^2[(2a+1)q+p]\alpha\beta^3\end{array}\right\} \\ v = \dfrac{1}{(2a+2)}\left\{\begin{array}{l}[(2a+1)q+p]\alpha^4 - 6(4a+3)[(2a+1)q+p]\alpha^2\beta^2 \\ +(4a+3)^2[(2a+1)q+p]\beta^4 + 4[(2a+1)p-(4a+3)q]\alpha^3\beta \\ -4(4a+3)[(2a+1)p-(4a+3)q]\alpha\beta^3\end{array}\right\}\end{array}\right\} \tag{19}$$

Substituting (19) in (2) we get,

$$\left.\begin{array}{l} x=\left\{\begin{array}{l}(p-q)\alpha^4+(4a+3)^2(p-q)\beta^4-6(4a+3)(p-q)\alpha^2\beta^2 \\ +4[-p-(4a+3)q]\alpha^3\beta+4(4a+3)[p+(4a+3)q]\alpha\beta^3\end{array}\right\} \\ y=\dfrac{1}{(2a+2)}\left\{\begin{array}{l}[2ap-(6a+4)q]\alpha^4+6(4a+3)[(6a+4)q-2ap]\alpha^2\beta^2 \\ +(4a+3)^2[2ap-(6a+4)q]\beta^4-4[(6a+4)p+2a(4a+3)q]\alpha^3\beta \\ +4(4a+3)[(4a+3)2aq+(6a+4)p]\alpha\beta^3\end{array}\right\}\end{array}\right\} \quad (20)$$

As our interest is in finding integer solutions, it is seen that replacing α by $(2a+2)M$ and β by $(2a+2)N$ in (20) and (4), the corresponding integer solutions to (1) are obtained and they are given below:

$$\left.\begin{array}{l} x=(2a+2)^4\left\{\begin{array}{l}(p-q)M^4+(4a+3)^2(p-q)N^4-6(4a+3)(p-q)M^2N^2 \\ +4[-p-(4a+3)q]M^3N+4(4a+3)[p+(4a+3)q]MN^3\end{array}\right\} \\ y=(2a+2)^3\left\{\begin{array}{l}[2ap-(6a+4)q]M^4+6(4a+3)[(6a+4)q-2ap]M^2N^2 \\ +(4a+3)^2[2ap-(6a+4)q]N^4-4[(6a+4)p+2a(4a+3)q]M^3N \\ +4(4a+3)[(4a+3)2aq+(6a+4)p]MN^3\end{array}\right\} \\ z=(2a+2)^2\left[M^2+(4a+3)N^2\right]\end{array}\right\} \quad (21$$

Note 2:

It is to be noted that, in addition to (16), 1 may also be represented as below:

(i). $$1=\frac{\left[(3-2a)+i\,3\sqrt{4a+3}\right]\left[(3-2a)-i\,3\sqrt{4a+3}\right]}{(2a+6)^2}$$

(ii). $$1=\frac{\left[(4a-1)+i\,4\sqrt{4a+3}\right]\left[(4a-1)-i\,4\sqrt{4a+3}\right]}{(4a+7)^2}$$

(iii). $$1=\frac{\left[(9-6a)+i\,9\sqrt{4a+3}\right]\left[(9-6a)-i\,9\sqrt{4a+3}\right]}{(6a+18)^2}$$

(iv). $$1=\frac{\left[(16a-4)+i\,16\sqrt{4a+3}\right]\left[(16a-4)-i\,16\sqrt{4a+3}\right]}{(16a+28)^2}$$

It is worth mentioning here that, by giving various integer values to a, p and q, one may obtain integer solutions to the corresponding biquadratic equation.

For illustration, the choices

$$a=1, p=8, q=1 \quad (22)$$

in (1) give

$$2(x^2+y^2)-3xy=71z^4 \quad (23)$$

Substituting (22) in (4), (8); (12), (14); and (21) the corresponding three sets of integer solutions to (23) are as follows:

Set 1:

$$x = 9\alpha^4 - 378\alpha^2\beta^2 + 441\beta^4 + 4\alpha^3\beta - 28\alpha\beta^3$$
$$y = 7\alpha^4 - 294\alpha^2\beta^2 + 343\beta^4 - 60\alpha^3\beta + 420\alpha\beta^3$$
$$z = \alpha^2 + 7\beta^2$$

Set 2:

$$x = 343r^4 - 294r^2s^2 + 7s^4 + 420r^3s - 60rs^3$$
$$y = 441r^4 - 378r^2s^2 + 9s^4 - 28r^3s + 4rs^3$$
$$z = 7r^2 + s^2$$

Set 3:

$$x = 1792M^4 - 75264M^2N^2 + 87808N^4 - 15360M^3N + 107520MN^3$$
$$y = 384M^4 - 16128M^2N^2 + 18816N^4 - 24064M^3N + 168448MN^3$$
$$z = 16M^2 + 112N^2$$

Further, it is observed that, by choosing suitably the values of a, p and q in (1), the solutions presented in [2, 6-9] are correspondingly obtained.

3. CONCLUSION

In this paper, an attempt has been made to obtain all integer solutions to the bi-quadratic equation $(a+1)(x^2+y^2)-(2a+1)xy = [p^2+(4a+3)q^2]\, z^4$. However, one may search for other choices of integer solutions to the above equation.

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