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# TAMELY NORMAL SPACES IN GENERAL TOPOLOGY

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**ABSTRACT**

*In the present paper, we introduce a weaker version of normal space called tamely normal. We will prove that tamely normal space is a property, which is implied by normal space. We prove that tamely normal space is a topological property and it is a hereditary property with respect to closed domain subspace. The relationships among normal, η*-normal, almost normal, quasi normal, mildly normal, π-normal are investigated. Moreover, we obtain some characterizations of tamely normal spaces.*

**KEY WORDS**: *η*-open sets; softly regular, η*-regular, almost-regular, η*-normal, π-normal, quasi normal, mildly normal, tamely normal and softly normal spaces*

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## 1. INTRODUCTION

In 1937, Stone [**17**] introduced the notion of semi-regular spaces and obtained their characterizations. In 1958, Kuratowski [**6**] introduced a generalization of closed sets, called regularly-open and regularly-closed sets in general topology. In 1963, Arhangel'skii [**1**] introduced the notion of k-space and proved that every open subspace of a regular k-space is a k-space. In 1968, the notion of a quasi normal space is a weaker form of normal space was introduced by Zaitsev [**20**]. In 1969, Singal and Arya [**14**] introduced a new class of separation axiom (named almost-regular space) in topological spaces which is weaker than regularity but it is equivalent to semi-regularity due to Stone [**17**]. In 1970, the concept of almost normality was introduced by Singal and Arya [**15**]. In 1973, the notion of mild normality was introduced by Shchepin [**13**] and Singal and Singal [ ] independently. In 1990, Lal and Rahman [**7**] further studied notion of quasi normal and mildly normal spaces. In 2008, π-normal topological spaces were introduced by Kalantan [**2**]. In 2015, M. C. Sharma and Hamant Kumar [**12**] introduced the concept of softly-normal spaces and obtained their characterizations. In 2018, H. Kumar and M. C. Sharma [**3**] introduced the concept of softly regular spaces and obtained some basic properties of softly regular spaces. In 2025, H. Kumar and J. Kumar [**4**] introduced the concepts of $\eta^*$-regular and $\eta^*$-normal spaces and obtained some basic properties of $\eta^*$-regular and $\eta^*$-normal spaces. Recently, H. Kumar and J. Kumar [**5**] further studied $\eta^*$-normal spaces and obtained some characterizations of $\eta^*$-normal spaces.

## 2. PRELIMINARIES

Throughout this paper, spaces (X, ℑ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** [**6, 17**] (resp. **regular closed [6, 17 ]**) if A = int(cl(A)) (resp. A = cl(int(A)) (or A subset A of a topological space (X, ℑ) is said to be **regularly-open** if it is the interior of its own closure or, equivalently, if it is the interior of some closed set. A subset A is said to be **regularly-closed** if it is the closure of its own interior or, equivalently, if it is the closure of some open set. Clearly, a set is regularly-open iff its complement is regularly-closed). The finite union of regular open sets is said to be **π-open [20]**. The complement of a π-open set is said to be **π- closed [20].**

**2.1 Remark**. Every regular open (resp. regular closed) set is π–open (resp. π-closed).

**2.2 Definition:** A subset A of a topological space (X, ℑ) is said to be **g-closed [8]** if cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
The complement of a g-closed set is said to be **g-open**.
The **generalized closure** of A is defined as the intersection of all g-closed sets in X containing A and is denoted by **cl*(A)**. The **generalized interior** of A is defined as the union of all g-open sets in X contained in A and is denoted by **int*(A)**.

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**2.3 Definition.** A subset A of a topological space (X, ℑ) is called **regular*-open** [**11**] (briefly r*-open) if A = int(cl*(A)). The complement of a regular*-open set is called **regular*-closed**. The union of all regular*-open sets of X contained in A is called **regular*-interior** of A and is denoted by **r*-int(A)**. The intersection of all regular*-closed sets of X containing A is called **regular*-closure of A** and is denoted by **r*-cl(A)**.

**2.4 Definition.** A subset A of a topological space (X, ℑ) is called **η*-open** [**9**] set if it is the union of regular*-open sets (r*-open sets).The complement of a η*-open set is called **η*-closed**. A subset A of a topological space (X, ℑ) is called **η*-Interior** of A if the union of all η*-open sets of X contained in A and is denoted by **η*-int(A)**. The intersection of all η*-closed sets of X containing A is called the **η*-closure** of A and is denoted by **η*-cl(A)**.

**2.5 Definition.** Let (X, τ) be a topological space and let A, B be any two subsets of X. A and B are said to be **weakly-separated** [**19**] if there exist open sets U and V such that $A \subset U$, $U \cap B = \phi$ and $B \subset V$, $V \cap A = \phi$. A and B are said to be **strongly-separated** [**10**] if there exist disjoint open sets U and V such that $A \subset U$, $B \subset V$.

**2.6 Definition.** A space X is said to be a **Urysohn space** [**18**] if every pair of distinct points x and y, there exist open sets U and V such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \phi$.

**2.7 Definition.** A topological space X is called **π-normal** [**2**] if for any two disjoint closed subsets A and B of X, one of which is π-closed, there exist two open disjoint subsets U and V of X such that $A \subset U$ and $B \subset V$.

**2.8 Definition.** A space X is said to be **η*-normal** [**4, 5**] if for every pair of disjoint η*-closed subsets A and B of X, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$.

**2.9 Definition.** A space X is said to be **quasi-normal** [**20**] if for every pair of disjoint π-closed subsets H, K, there exist disjoint open sets U, V of X such that $H \subset U$ and $K \subset V$.

**2.10 Definition.** A space X is said to be **softly-normal** [**12**] if for every pair of disjoint subsets A, B of X, one of which is π-closed and the other is regularly-closed, there exist disjoint open sets U, V of X such that $A \subset U$ and $B \subset V$.

**2.11 Definition.** A topological space X is said to be **almost normal** [**15**] if for every pair of disjoint sets A and B, one of which is closed and the other is regularly closed, there exist open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$.

**2.12 Definition.** A space X is said to be **mildly normal** [7, **13, 16**] if for every pair of disjoint regularly closed subsets A and B of X, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$.

**2.13 Definition.** A space (X, τ) is said to be **softly regular** [**3**] if for each π–closed set A and each point $x \in X - A$, there exist disjoint open sets U, V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

**2.14 Definition.** A space (X, τ) is said to be **almost regular** [**14**] if for every regularly closed set A and a point $x \notin A$, there exist open sets U, V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

**2.15 Definition.** A space (X, τ) is said to be **weakly regular [14]** if every weakly point x and every regularly-open set U containing x, there is an open set V such that $x \in V \subset cl(V) \subset U$.

**2.16 Definition.** A space (X, τ) is said to be **semi-regular** [**17**] if for each point x of the space and each open set U containing x, there is an open set V such that $x \in V \subset int(cl(V)) \subset U$.

**By the definitions stated above, we have the following diagram:**

regular ⇒ η*-regular ⇒ softly regular ⇒ almost regular ⇒ weakly regular

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**Where none of the implications is reversible as can be seen from the following examples**

**2.17 Example.** Let X = {a, b, c} and $\Im$ = {$\phi$, {a}, {b, c}, X}. Consider the closed set {b, c} and a point 'a' such that a $\notin$ {b, c}. Then {b, c} and {a} are open sets such that {b, c} $\subset$ {b, c}, a $\in$ {a} and {b, c} $\cap$ {a} = $\phi$. Similarly for the closed set {a} and a point 'c' such that c $\notin$ {a}. Then there exist open sets {a} and {b, c} such that {a} $\subset$ {a}, c $\notin$ {b, c} and {a} $\cap$ {b, c} = $\phi$. It follows that (X, $\Im$) is regular as well as $\eta^*$-regular space. It is also almost regular.

**2.18 Example.** Let X = {a, b, c} and $\Im$ = {$\phi$, {a}, {b}, {a, b}, X}. If we take a point a and an open set V = {a}, then cl(V) = {a, c} and a regularly-open set U = X. So by the definition of weakly regular space x $\in$ V $\subset$ cl(V) $\subset$ U, where V be an open set and U be a regularly-open set such that a $\in$ {a} $\subset$ {a, c} $\subset$ X. Hence (X,) is weakly regular. If we take a point a and a regularly-closed set A = {b, c} does not containing the point a, there donot exist disjoint open sets containing a point a and a regularly-closed set A = {b, c}. Hence (X, $\Im$) is not almost-regular.

## 3. TAMELY NORMAL SPACES

**3.1 Definition.** A topological space X is said to be **tamely normal** if for every pair of disjoint sets A and B, one of which is closed and the other is $\eta^*$-closed, there exist open sets U and V such that A $\subset$ U, B $\subset$ V and U $\cap$ V = $\phi$.

By the definitions stated above, we have the following diagrams:

normal $\Rightarrow$ tamely normal $\Rightarrow$ $\pi$-normal $\Rightarrow$ almost normal $\Rightarrow$ softly normal $\Rightarrow$ mildly normal

and

normal $\Rightarrow$ tamely normal $\Rightarrow$ $\pi$-normal $\Rightarrow$ quasi normal $\Rightarrow$ softly normal $\Rightarrow$ mildly normal

and

normal $\Rightarrow$ tamely normal $\Rightarrow$ $\eta^*$-normal $\Rightarrow$ quasi normal $\Rightarrow$ softly normal $\Rightarrow$ mildly normal

**Where none of the implications is reversible as can be seen from the examples in [2, 3, 4, 7, 12, 15, 16, 20]**

**3.2 Remark.** $\pi$-normal spaces and $\eta^*$-normal spaces are independent properties.

**3.3 Theorem.** For a space X, the following are equivalent:
(i) X is tamely normal.
(ii) For every $\eta^*$-closed set A and every open set B with A $\subset$ B, there exists an open set U such that A $\subset$ U $\subset$ cl(U) $\subset$ B.
(iii) For every closed set A and every $\eta^*$-open set B with A $\subset$ B, there exists an open set U such that A $\subset$ U $\subset$ cl(U) $\subset$ B.
(iv) For every pair consisting of disjoint sets A and B, one of which is $\eta^*$-closed and the other is closed, there exist open sets U and V such that A $\subset$ U, B $\subset$ V and cl(U) $\cap$ cl(V) = $\phi$.

**Proof.**
(i) $\Rightarrow$ (ii). Assume (i). Let A be any $\eta^*$-closed set and B be any open set such that A $\subset$ B. Then A $\cap$ (X $-$ B) = $\phi$, where (X $-$ B) is closed. Then there exist disjoint open sets U and V such that A $\subset$ U and (X $-$ B) $\subset$ V. Since U $\cap$ V= $\phi$, then cl(U) $\cap$ V = $\phi$. Thus cl(U) $\subset$ (X $-$ V) $\subset$ (X $-$ (X $-$ B)) = B. Therefore, A $\subset$ U $\subset$ cl(U) $\subset$ B.

(ii) $\Rightarrow$ (iii). Assume (ii). Let A be any closed set and B be any $\eta^*$-open set such that A $\subset$ B. Then, (X $-$ B) $\subset$ (X $-$ A), where (X $-$ B) is $\eta^*$-closed and (X $-$ A) is open. Thus by (ii), there exists an open set W such that (X $-$ B) $\subset$ W $\subset$ cl(W) $\subset$ (X $-$ A). Thus A $\subset$ (X $-$ cl(W)) $\subset$ (X $-$ W) $\subset$ B. So, let U = (X $-$ cl(W)), which is open and since W $\subset$ cl(W), then (X $-$ cl(W) $\subset$ (X $-$ W). Thus U $\subset$ (X $-$ W), hence cl(U) $\subset$ cl(X $-$ W) = (X $-$ W) $\subset$ B.

(iii) $\Rightarrow$ (iv). Assume (iii). Let A be any closed set and B be any $\eta^*$-closed set with A $\cap$ B = $\phi$. Then A $\subset$ (X $-$ B), where (X $-$ B) is $\eta^*$-open. By (iii), there exists an open set U such that A $\subset$ U $\subset$ cl(U) $\subset$ (X $-$ B). Now, cl(U) is closed. Applying (iii) again we get an open set W such that A $\subset$ U $\subset$ cl(U) $\subset$ W $\subset$ cl(W) $\subset$ (X $-$ B). Let V = (X $-$ cl(W)), then V is open set and B $\subset$ V. We have (X $-$ cl(W)) $\subset$

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(X − W), hence V ⊂ (X − W), thus cl(V) ⊂ cl(X − W) = (X − W). So, we have cl(U) ⊂ W and cl(V) ⊂ (X − W). Therefore cl(U) ∩ cl(V) = ϕ.

(iv) ⇒ (i) is clear.

Using **Theorem 3.3,** it is easy to show the following theorem, which is a Urysohn's Lemma version for tamely normal space. A proof can be established by a similar way of the normal case.

**3.4 Theorem.** A space X is tamely normal if and only if for every pair of disjoint closed sets A and B, one of which is η*-closed and the other is closed, there exists a continuous function f on X into [0, 1], with its usual topology, such that f(A) = {0} and f(B) = {1}.

It is easy to see that the inverse image of a η*-closed set under an open continuous function is η*-closed and the inverse image of a π-closed set under an open continuous function π-closed. We will use that in the next theorem.

**3.5 Theorem.** Let X is a tamely normal space and f : X→ Y is an open continuous injective function. Then f(X) is a lamely normal space.
**Proof.** Let A be any η*-closed subset in f(X) and let B be any closed subset in f(X) such that A ∩ B = ϕ. Then $f^{-1}(A)$ is a η*-closed set in X, which is disjoint from the closed set $f^{-1}(B)$. Since X is lamely normal, there are two disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Since f is 1-1 and open, result follows.

**3.6 Corollary.** Tamely normal space is a topological property.

**3.7 Lemma.** Let M be a closed domain subspace of a space X. If A is an open set in X, then A ∩ M is open set in M.

**3.8 Theorem.** A closed domain subspace of a tamely normal space is tamely normal.
**Proof.** Let M be a closed domain subspace of a tamely normal space X. Let A and B be any disjoint closed sets in M such that A is closed and B is η*-closed. Then, A and B are disjoint closed sets in X such that A is closed and B is η*-closed in X. By tamely normal space X, there exist disjoint open sets U and V of X such that A ⊂ U and B ⊂ V. By the **Lemma 3.7,** we have U ∩ M and V ∩ M are disjoint open sets in M such that A ⊂ U ∩ M and B ⊂ V ∩ M. Hence, M is tamely normal subspace.

Since every closed and open (clopen) subset is a closed domain, then we have the following corollary.

**3.9 Theorem.** A clopen subspace of a tamely normal space is tamely normal.

**3.10 Theorem**. Every η*-regular, semi-regular space is regular.
**Proof**. Suppose X is a η*-regular, semi-regular space. Let M be any open set containing a point x ∈ X. Since X is semi-regular, there exists an open set U such that x ∈ U ⊂ int(cl(U)) ⊂ M. Again, since U is an open neighbourhood of x and X is η*-regular, there is an open set V such that x ∈ V ⊂ cl(V) ⊂ int(cl(U)). Hence x ∈ V ⊂ cl(V) ⊂ M. Thus X is regular.

**Conclusion**: In this paper, we study a new class of generalized normal space is called tamely normal space which is weaker than normal space. The relationships among normal, tamely normal, η*-normal, quasi normal, softly normal, almost normal, π-normal and mildly normal spaces are investigated. Some of basic properties and characterizations of tamely normal spaces are obtained. This idea can be extended to topological ordered, bitopological, bitopological ordered and fuzzy topological spaces etc. We also define separation axioms by using $\eta^*$-open sets such as

**Conflict of Interest:** We certify that this work is original, have never been published before, and is not being considered for publication anywhere else at this time. This publication is free from any conflicts of interest. As the Corresponding Author, I certify that each of the listed Authors has read the paper and given their approval for publication.

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