ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 4| April 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

FORMULATING EDGE COVERAGE CONSTRAINTS AND ANALYZING CONNECTIVITY CONDITIONS ON BUTTERFLY GRAPHS

Megha Chandraprakash Gaikwad[1]
[1]*Assistant Professor, Shri Sheth Muralidharji Mansingaka Arts, science and commerce College, Pachora , Dist. Jalgaon*

Article DOI: https://doi.org/10.36713/epra27073
DOI No: 10.36713/epra27073

ABSTRACT
The Weighted Connected Vertex Cover (W-CVC) problem is a significant combinatorial optimization problem that integrates vertex coverage, connectivity, and weight minimization. This paper focuses on the formulation of edge coverage constraints and the analysis of connectivity conditions specifically on butterfly graphs, a structured class of graphs widely used in parallel computing and communication networks. By exploiting the inherent structural properties of butterfly graphs, particularly the presence of an articulation vertex, the study presents a simplified and efficient framework for solving the W-CVC problem. The work highlights how structural decomposition reduces computational complexity and enables the derivation of a closed-form solution.

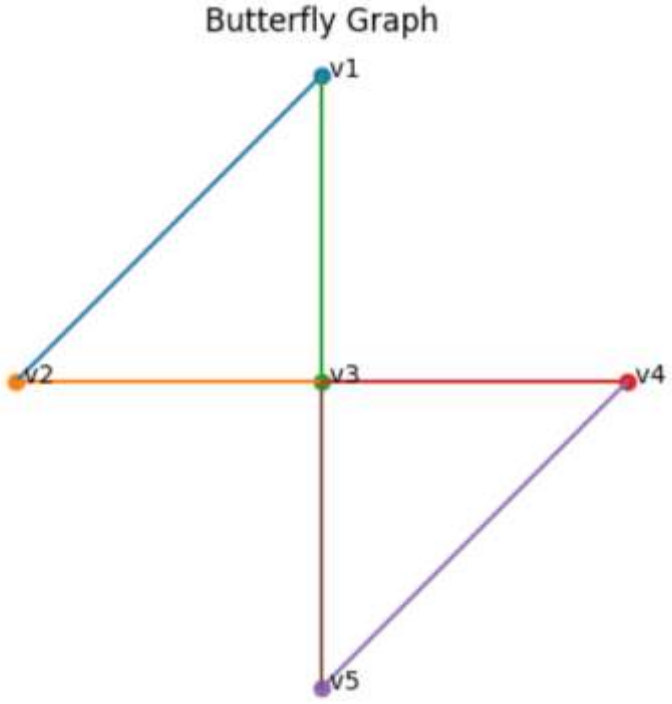
KEYWORDS: *Vertex Cover, Connected Vertex Cover, Butterfly Graph, Edge Coverage, Connectivity Constraints, Graph Optimization*

INTRODUCTION
Graph theory plays a crucial role in various domains such as computer science, operations research, and network design. Among the fundamental problems in graph theory is the Vertex Cover Problem, which involves selecting a subset of vertices such that every edge in the graph is incident to at least one selected vertex. [3]

The Connected Vertex Cover (CVC) problem extends this concept by requiring that the selected vertices form a connected subgraph. When weights are assigned to vertices, the problem becomes the Weighted Connected Vertex Cover (W-CVC) problem, which is computationally more challenging and belongs to the class of NP-hard problems.

This paper focuses on the formulation of edge coverage constraints and the analysis of connectivity conditions in butterfly graphs, where structural properties allow simplification of the problem.[5]

Butterfly Graph Structure
A butterfly graph is a structured graph consisting of two triangles sharing a common vertex.



ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 4| April 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

A Butterfly Graph Bn is a layered graph with:
- n+1 levels (from 0 to n) [2]
- each level contains 2^n vertices
- Total vertices: $(n+1)^2$

Case Analysis:

Case 1: $x3 = 1$

Both triangles are connected through v3v_3v3, The selected vertices form a connected subgraph.

Case 2: $x3 = 0$

The graph splits into two disconnected components connectivity condition may fail.

For any edge $(vi, vj) \in E(v_i, v_j) \backslash in\ E(vi, vj) \in E$:

$$xi + xj \geq i\ +\ xi + xj \geq 1$$

Connectivity Condition Simplification

To guarantee connectivity:

$$x3 = 1$$

Deriving Constraints:

The W-CVC problem on butterfly graphs can be expressed as:

$minZ = i = 1\sum 5wixi$

Subject to

- Edge coverage constraints
- Connectivity condition $x3 = 1$
- Binary constraints $xi \in \{0,1\}$

The edge coverage constraints form the foundational requirement of the Weighted Connected Vertex Cover (W-CVC) problem. These constraints ensure that every edge in the graph is incident to at least one selected vertex, thereby guaranteeing that no edge remains uncovered.[4]

In a graph $G = (V, E)$, each edge connects two vertices. To "cover" an edge means selecting at least one of its endpoints. This requirement is translated into a mathematical constraint using binary decision variables. For each vertex vi, a variable $xi \in \{0,1\} x_i \backslash in \backslash \{0,1\backslash\} xi \in \{0,1\}$ is defined, where $xi = 1 x_i\ =\ 1 xi = 1$ indicates that the vertex is included in the vertex cover.[2]

For every edge $(vi, vj) \in E(v_i, v_j) \backslash in\ E(vi, vj) \in E$, the edge coverage condition is expressed as:

$$xi + xj \geq 1 x_i\ +\ x_j \backslash geq\ 1 xi + xj \geq 1$$

This inequality ensures that at least one of the two vertices connected by the edge is selected. If both $xix_ixi\ and\ xjx_jxj$ were zero, the edge would remain uncovered, which violates the definition of a vertex cover.

An important characteristic of edge coverage constraints is that they are uniform and structure-independent. The same formulation applies to all types of graphs, whether simple or complex. However, while these constraints ensure coverage, they do not guarantee connectivity. The connectivity requirement is handled separately through additional conditions.

Thus, edge coverage constraints ensure the feasibility of the vertex cover, forming the first essential component of the W-CVC problem. When combined with connectivity constraints and the objective function, they complete the mathematical formulation of the optimization problem. [1]

In the case of butterfly graphs, the structure consists of two interconnected triangles. Each triangle contributes three edges, resulting in a total of six edges. Accordingly, six edge coverage constraints are formulated—one for each edge. These constraints are applied locally, meaning each edge is treated independently, without considering the overall structure of the graph. [6]

Model Reduction Using Structure: With$x3 = 1$, constraints simplify to:

$$x1 + x2 \geq 1$$
$$x4 + x5 \geq 1$$

This decomposes the problem into two independent sub-problems.

Computational Implications:

The structural properties of butterfly graphs lead to:

Reduction in search space elimination of global connectivity checks constant-time solution.

Optimal Solution

$$Z *= w3 + min\ (w1, w2) + min\ (w4, w5) Z\hat{}\ *$$
$$=\ w_3\ +\ \backslash min(w_1, w_2)$$
$$+\ \backslash min(w_4, w_5) Z *$$
$$= w3 + min(w1, w2) + min(w4, w5)$$

The optimal solution is expressed as

$$Z *= w3 + \min(w1, w2) + \min(w4, w5)\ Z^* =$$
$$w_3 + \backslash min(w_1, w_2) + \backslash min(w_4, w_5) Z *$$
$$= w3 + min(w1, w2) + min(w4, w5),$$

which indicates a structured selection process. In this formulation, the value $w3w_3w3$ is always included in the final solution, suggesting that it represents a mandatory component or a fixed contribution that cannot be avoided. Alongside this, the solution involves choosing the minimum value from two separate pairs:

$(w1, w2)(w_1, w_2)(w1, w2)\ and\ (w4, w5)(w_4, w_5)(w4$ $, w5)$.

This means that, within each pair, only the smaller weight is selected, ensuring that the total value of $Z * Z\hat{}\ * Z *$ is minimized.

This structure reflects an optimization strategy where decisions are made independently across different groups of variables. Each pair represents a mutually exclusive choice, and selecting the minimum in each case guarantees the lowest possible contribution from that group. Since these choices do not interfere with one another, the overall optimal solution is obtained by combining the best local decisions (the minima of each pair) with the fixed component w_3. This leads to an efficient and globally optimal result without requiring exhaustive comparison of all possible combinations.

DISCUSSION

The formulation of edge coverage constraints remains consistent across graph types. However, the analysis of connectivity conditions varies significantly depending on graph structure.

In butterfly graphs:

Assume there exists a distinguished vertex $v *\in V$ such that:

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 4| April 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

$$\forall u \in V \setminus \{v*\}, (u, v*) \in E$$
$$forall\ u\ \in V\ \setminus \{v^{*}\},$$
$$quad\ (u, v\hat{}*)\ in\ E \forall u \in V \setminus \{v*\}, (u, v*) \in E$$

This means $v * v\hat{} * v *$ is a universal (hub) vertex, ensuring connectivity of the graph:

$\kappa(G) = 1$, and removal of $v *$ disconnects $(G) = 1$,

$G\kappa(G) = 1$, and removal of v∗ disconnects G

Connectivity is ensured by a single vertex, problem reduces to local decisions.

Complexity decreases from exponential to constant time, this demonstrates the importance of graph topology in optimization problems.

CONCLUSION

This paper presented a detailed formulation of edge coverage constraints and analysis of connectivity conditions for butterfly graphs in the context of the W-CVC problem. The study shows that structural properties, particularly the presence of an articulation vertex, simplify the connectivity requirement and enable efficient problem decomposition. The findings highlight how exploiting graph structure can transform a computationally hard problem into an efficiently solvable one.

REFERENCES

1. *R. M. Karp, "Reducibility among combinatorial problems," 1972.*
2. *M. Garey and D. Johnson,* Computers and Intractability, *1979.*
3. *F. Gavril, "Algorithms for minimum coloring and clique problems," 1977.*
4. *S. Guha and S. Khuller, "Approximation algorithms," 1998.*
5. *X. Zhang et al., "Heuristic approaches for vertex cover," 2014.*