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FUNDAMENTAL CONCEPTS OF FUZZY TOPOLOGY

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ABSTRACT
In this paper, we study some basic terms of fuzzy topology such as indiscrete and discrete topology, comparison of topologies, closed sets, door space, intersection and union of topologies, interior, exterior and boundary of a set, and discuss these terms by some examples.

1. INTRODUCTION

The concept of fuzzy sets and fuzzy operations was introduced by Lotfi A. Zadeh [**4**]. It is an extension of the notion of classical or crisp set theory due to Cantor. It is an essential tool for formulating the mathematical information of imprecise nature. The notion of a fuzzy topology was introduced by C. L. Chang [**1**] in 1968. The notion of fuzzy sets plays a very significant roie in the study of fuzzy topology. It is an extension of the concept of general topology. Subsequently many researchers have been worked in this area and related area meaning full and powerful representation of measurement of uncertainty and meaningful representation off Ve concept Express in natural language.

In this paper, we study some basic terms of fuzzy topology such as indiscrete and discrete topology, comparison of topologies, closed sets, door space, intersection and union of topologies, interior, exterior and boundary of a set.

2. CRISP SET THEORY

In the ordinary or crisp or classical set theory, a set is a "well defined" collection of distinct objects. By the property "well defined" we mean that there is a given rule by means of what it is possible to know whether a particular element (or object or member) is contained in the collection or not. Let X be a set and A be a subset of X. We write $x \in A$ if an element x of X is a member of A and $x \notin A$ if x of X is not a member of A.

Membership in a subset A of X is based on two valued logic true (or 1) and false (or 0) and can be restated in terms of characteristic function (or membership function) μ_A from X to {0, 1} i. e.

$$\mu_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Let us consider a finite set X = {a, b, c, d, e, f} and a subset A = {a, c, e}. The set A can be represented by the set of pairs i.e.

$A = \{(x, \mu_A(x)) : x \in X\}$
$= \{(a, 1), (b, 0), (c, 1), (d, 0), (e, 1), (f, 0)\}$.

Where μ_A is the known characteristic function and
$\mu_A : X \to \{0, 1\}$

Thus the subset A of a set X can be characterized by a characteristic function which associates with each x its grade of membership $\mu_A(x)$.

3. FUZZY SET THEORY

As defined in the previous section, the characteristic function of a crisp set assigns a value of either 1 or 0 to each individual in the universal set, thereby discriminating between members and nonmembers of the crisp set under consideration. This function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Then the larger values denote higher degrees of set membership. Such a function is called a membership function, and the set defined by it a fuzzy set.

The most commonly used range of values of membership functions is the unit interval [0, 1]. In this case, each membership function maps elements of a given universal set X, which is always a crisp set, into real numbers in [0, 1].

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The membership functions of a fuzzy set A is denoted by μ_A' that is

$\mu_A : X \to [0, 1]$.

Let X = {x} be a collection of objects, then a fuzzy set A in X is a set of ordered pairs.

$A = \{(x, \mu_A(x)) : x \in X\}$

Where $\mu_A(x)$ is called the "membership grade" of x in A and each pair $(x, \mu_A(x))$ is called singleton.

A degree of zero means that the values is not in the set, and a degree of one means that the values is completely representative of the set.

Consider X = {a, b, c, d} and A : X → I defined by A(a) = 0.0, A(b) = 0.7, A(c) = 0.4 and A(d) = 1.0. Then, the fuzzy set A on X is written as

A = {(a, 0.0), (b, 0.7), (c, 0.4), (d, 1.0)}.

4. STANDARD OPERATION ON FUZZY SETS

4.1 Complement of a fuzzy set

Let A be a fuzzy subset of a nonempty set X. The complement of A is denoted by A^c and defined as

$A^c(x) = 1 - A(x), \forall x \in X$.

Elements of X for which $A(x) = A^c(x)$ are called equilibrium point of A. Clearly, membership grades of equilibrium point are 0.5.

4.2 Union of two fuzzy sets

Let A and B are two fuzzy subsets of a nonempty set X. The union of A and B is denoted by $A \cup B$ and defined as:

$$(A \cup B)(x) = \max[A(x), B(x)], \forall x \in X.$$

4.3 Intersections of two fuzzy sets

Let A and B are two fuzzy subsets of a nonempty set X. The intersection of A and B is denoted by $A \cap B$ and defined as:

$$(A \cap B)(x) = \min[A(x), B(x)], \forall x \in X.$$

5. FUZZY TOPOLOGY

5.1 Definition. Let X be a non-empty set and I^X be the family of all fuzzy subsets on X. A family $\mathbf{F} \subset I^X$ of fuzzy subset is called a fuzzy topology for X if it satisfies the following three axioms:

$[F_1]$: $0 = \mu_\phi \in \mathbf{F}$ and $1 = \mu_X \in \mathbf{F}$.

i.e. the constant functions 0 (empty fuzzy set) and 1 (whole fuzzy set) are in **F**.

$[F_2]$ for all $\alpha, \beta \in \mathbf{F} \Rightarrow \alpha \cap \beta \in \mathbf{F}$.

i.e. the intersection (infimum) of any two fuzzy sets in **F** is in **F**.

$[F_3]$: If $\{\mu_i : i \in \Delta\} \subset \mathbf{F}$, where Δ denotes an index set, then $\vee \mu_i \in \mathbf{F}$.

Then **F** is called a fuzzy topology for X and the pair (X, **F**) is called a fuzzy topological space (briefly fts). The members of **F** are called fuzzy open sets.

Let (X, **F**) be topological space. A subset F of X is said to be fuzzy closed if and only if its complement F^c is fuzzy open.

5.2 Definition. Let **F** be a fuzzy topology on X, and A be a fuzzy set in X. Then the fuzzy set A is said to be
(i) fuzzy open if $A \in \mathbf{F}$.

(ii) Fuzzy closed if $A^c \in \mathbf{F}$.

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(iii) fuzzy clopen fuzzy if A and $A^c \in \mathbf{F}$.

5.3 Example. Let X = {a, b, c,}, A = {(a, 0.8), (b, 0.9), (c, 0.7)}, B = {(a, 0.6), (b, 0.5), (c, 0.4)}, C = {(a, 0.3), (b, 0.3), (c, 0.2)} and F = {0, A, B, C, 1}.
Now,
A ∧ B = {(a, 0.6), (b, 0.5), (c, 0.4)} = B∈ **F**,
A ∧ C = {(a, 0.3), (b, 0.3), (c, 0.2)} = B∈ **F**,
B ∧ C = {(a, 0.3), (b, 0.3), (c, 0.2)} = B∈ **F**,
A ∨ B = {(a, 0.8), (b, 0.9), (c, 0.7)} = A∈ **F**,

A ∨ C = {(a, 0.8), (b, 0.9), (c, 0.7)} = A∈ **F**,

B ∨ C = {(a, 0.6), (b, 0.5), (c, 0.4)} = B∈ **F**.

Also

A ∨ B ∨ C = {(a, 0.6), (b, 0.9), (c, 0.7)} = A ∈ **F.**

It is obvious 0, l ∈ **F**.

Therefore, (X, **F**) is a fuzzy topological space.

5.4 Example. Let X = {a, b} and B, C, D are fuzzy subsets in A, where
A = {(a, 0.9), (b, 0.9)}, B = {(a, 0.0), (b, 0.7)}, C = {(a, 0.8), (b, 0.0)}, D = {(a, 0.8), (b, 0.7)}.
The fuzzy topology defined on A is **F** = {ϕ, A, B, C, D}

5.5 Example. Let X = {a, b}. Let A be a fuzzy set on X defined as A(a) = 0.5, A(b) = 0.4. Then **F** = $\{\bar{0}, A, \bar{I}\}$ is a fuzzy topology. (X, **F**) is a fuzzy topological space. $\bar{0}(a) = 0$, ∀ a ∈ X and $\bar{1}$(a) = 1 ∀ a ∈ X.

5.6 Example. Let X = {x, y}, α = {(x, 0.6), (y, 0.3)}, β = {(x, 0.2), (y, 0.7)}, and **F** = {ϕ, X, α, β, α ∧ β, α ∨ β}.

(α ∧ β) (x) = int{0.6, 0.2} = 0.2,

(α ∧ β) (y) = int{0.3, 0.7} = 0.3,

So, (α ∧ β) = {(x, 0.3), (y, 0.3)}.

Moreover, (α ∨ β)(x) = sup{0.6, 0.2} = 0.6,

(α ∨ β)(y) = sup{0.3, 0.7} = 0.7.

So, (α ∨ β) = {(x, 0.6), (y, 0.7)}.

Note that (X ∨ α)(x) = sup{1, 0.6} = 1,

(X ∨ α)(y) = sup{0.3, 1} = 1.

So, (X ∨ α) = X.

Similarly, α ∧ X = α, ϕ ∧ α = ϕ and ϕ ∨ α = α.

Now, the collection

F = {ϕ, X, α, β, α ∧ β, α ∨ β} is a fuzzy topology. So, we have that ϕ, X, α, β, α ∧ β, α ∨ β are fuzzy open sets in fuzzy topology.

Moreover, the fuzzy sets α^c = {(x, 0.4), (y, 0.7)}, β^c = {(x, 0.8), (y, 0.3)}, ϕ and X are fuzzy closed sets in **F**.

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5.7 Indiscrete fuzzy topology

Let X be a non-empty set. First of all, we define two special fuzzy sets:

(i) Zero fuzzy set (Completely empty – membership 0 everywhere)

$0_X(x) = 0 \ \forall \ x \in X.$

(ii) Unit fuzzy set (Completely full – membership 1 everywhere)

$1_X(x) = 1 \ \forall \ x \in X.$

Then the Collection

$\mathbf{I} = \{0_X, 1_X\}$ is called an indiscrete fuzzy topology on X. The pair (X, $\mathbf{I}$) is called an indiscrete fuzzy topological space.

Check three axioms:

(1) $0_X, 1_X \in X.$
(2) Finite intersection (infimum):

$\inf\{0_X, 1_X\} = 0_X \in \mathbf{I}.$

(3) Arbitrary union (supremum):

$\sup\{0_X, 1_X\} = 1_X \in \mathbf{I}.$

So all axioms are satisfied.

Hence it is a valid fuzzy topology.

So this is the coarsest possible fuzzy topology, exactly like the classical indiscrete topology.

5.8 Example. Let X = {a, b, c}.

Define:

(i) $0_X = \{(a, 0), (b, 0), (c, 0)\}.$

(ii) $1_X = \{(a, 1), (b, 1), (c, 1)\}$

Then, $\mathbf{I} = \{0_X, 1_X\}$ is the indiscrete fuzzy topology on X.

5.9 Fuzzy Discrete Topology

Let X be a nan empty set. Let I^X be note the family of all fuzzy subsets of X (i.e., all functions $A : X \to [0, 1]$)

Define

$\mathbf{D} = I^x = \{A : X \to [0, 1]\}$

Then $\mathbf{D}$ is called the fuzzy discrete topology on X. The pair (X, $\mathbf{D}$) is called a fuzzy discrete topological space.

5.10 Remark.
(i) Every fuzzy set on X is fuzzy open.
(ii) No restriction on openness.
(iii) Any membership function is allowed.
(iv) This is the finest possible fuzzy topology.
(v) Every fuzzy set is both open and closed.

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(vi) Every fuzzy set is clopen.

5.11 Comparison of fuzzy topologies

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two fuzzy topologies for X. If the inclusion relation $\mathbf{F}_1 \subset \mathbf{F}_2$ holds, we say that $\mathbf{F}_2$ is finner (stronger or bigger) than $\mathbf{F}_1$ and $\mathbf{F}_1$ is coarser (smaller, weaker) than $\mathbf{F}_2$.

5.11 Examples

(i) Indiscrete fuzzy topology is coarser topology and discrete fuzzy topology is stronger topology.

(ii) Every fuzzy topology on X is contained in the fuzzy discrete topology.

S. No.	Type	Open sets	Special Feature	Clopen sets	Structure
1.	Fuzzy Discrete Topology	All fuzzy sets	Finest topology	All fuzzy sets	Maximum
2.	Fuzzy indiscrete topology	Only 0_X, 1_X	Coarsest topology	Only 0_X, 1_X	Minimum

5.12 Intersection and Union of Topologies

(i) Intersection of Two Fuzzy Topologies

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two topologies for a non-empty set X. Then intersection of two fuzzy topologies on X is again a fuzzy topology on X.

5.13 Example. Let X = {a, b}. Take $\mathbf{F}_1 = \{0_X, 1_X, A\}$ and $\mathbf{F}_2 = \{0_X, 1_X, B\}$, where A and B are fuzzy sets.

Then:

$\mathbf{F}_1 \cap \mathbf{F}_2 = \{0_X, 1_X\}$ is also a fuzzy topology, called the indiscrete fuzzy topology.

If $\mathbf{F}_1$ and $\mathbf{F}_2$ are fuzzy topologies on X, then $\mathbf{F}_1 \cap \mathbf{F}_2$ is also a fuzzy topology on X.

5.14 Theorem. The intersection of two fuzzy topologies on X is again a fuzzy topology on X.

Proof. First we verify Chang's axioms:

Let $\mathbf{F}_1$, and $\mathbf{F}_1$ be two fuzzy topologies on X.

We have two show $\mathbf{F} = \mathbf{F}_1 \wedge \mathbf{F}_2$ is a fuzzy topology on X.

[F_1]: Contains 0_X and 1_X :

Since $\mathbf{F}_1$ and $\mathbf{F}_1$ are two fuzzy topologies, $0_X, 1_X \in \mathbf{F}_1$ and $0_X, 1_X \in \mathbf{F}_2$.

Hence $0_X, 1_X \in \mathbf{F}_1 \wedge \mathbf{F}_2 = \mathbf{F}$.

[F_2]: Closed under finite fuzzy intersection (Infima)

Let A, B $\in \mathbf{F}$.

Then A, B $\in \mathbf{F}_1$ and A, B $\in \mathbf{F}_2$.

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Since both are fuzzy topologies A, B $\in \mathbf{F}_1$ and A, B $\in \mathbf{F}_2$.

Thus, A $\cap$ B $\in \mathbf{F}$.

[F_3]: Closed under arbitrary fuzzy union (suprema)

Let $A_i \in \mathbf{F}$, where i $\in \Delta$, Δ is an index set.

Then each $A_i \in \mathbf{F}_1$ and $A_i \in \mathbf{F}_2$. Because both $\mathbf{F}_1$ and $\mathbf{F}_2$ are fuzzy topologies,

$\frac{\sup Ai}{i \in \Delta} \in \mathbf{F}_1$ and $\frac{\sup Ai}{i \in \Delta} \in \mathbf{F}_2$.

Therefore, $\frac{\sup Ai}{i \in \Delta} \in \mathbf{F}_1 \wedge \mathbf{F}_2 = \mathbf{F}$.

Since all three axioms hold, so $\mathbf{F}_1 \wedge \mathbf{F}_2 = \mathbf{F}$ is again a fuzzy topology.

(ii) Union of Two Fuzzy Topologies

The union of two fuzzy topologies on X need not be a fuzzy topology.

A fuzzy topology must be

(a) Closed under arbitrary fuzzy union (suprema)

(b) Closed under finite fuzzy intersection (Infima)

Even if $\mathbf{F}_1$ and $\mathbf{F}_2$ each satisfy this separately; their union may fail closure under intersection.

(iii) Example. Let $X = \{a, b\}$. Define fuzzy sets:

(a) $0_X = \{(a, 0), (b, 0)\}$

(b) $1_X = \{(a, 1), (b, 1)\}$

Let A = $\{(a, 1), (b, 0)\}$ and B = $\{(a, 0), (b, 1)\}$.

Now define two fuzzy topologies:

$\mathbf{F}_1 = \{0_X, 1_X, A\}$ and $\mathbf{F}_2 = \{0_X, 1_X, B\}$.

Each is a fuzzy topology.

Take their union:

$\mathbf{F} = \mathbf{F}_1 \wedge \mathbf{F}_2 = \{0_X, 1_X, A, B\}$.

Now compute the fuzzy intersection:

A $\cap$ B = min$\{A, B\}$ = $\{(a, 0), (b, 0)\} = 0_X$.

But consider instead slightly modified fuzzy sets:

Let A = $\{(a, 0.7), (b, 0.2)\}$ and B = $\{(a, 0.3), (b, 0.8)\}$. Again form

$\mathbf{F}_1 = \{0_X, 1_X, A\}$ and $\mathbf{F}_2 = \{0_X, 1_X, B\}$.

Then A $\cap$ B = $\{(a, 0.3), (b, 0.2)\}$

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This fuzzy set is not in $\mathbf{F}_1 \cup \mathbf{F}_2$.

Hence $\mathbf{F}$ is not closed under finite intersection,

so, $\mathbf{F}_1 \cup \mathbf{F}_2$ is not a fuzzy topology.

5.15 Remark.

(i) The intersection of two fuzzy topologies is again a fuzzy topology.

(ii) The union of two fuzzy topologies generally not a fuzzy topology.

5.16 Door Space.

A fuzzy topological space (X, $\mathbf{F}$) is called a fuzzy door spare if every fuzzy subset of X is either fuzzy open or fuzzy closed (or both).

Symbolically, for every fuzzy set A,

$A \in \mathbf{F}$ or $A^c \in \mathbf{F}$.

Types of fuzzy door spaces

(i) Fuzzy open door space.

Every fuzzy set is fuzzy open.

That is,

$\mathbf{F} = I^X$ (all fuzzy sets).

This is exactly the fuzzy discrete topology.

(ii) Fuzzy closed door space

Every fuzzy set is fuzzy closed.

That is

$\mathbf{F} = \{0_X, 1_X\}$.

(iii) Strong fuzzy door space

Every fuzzy set is either open or closed but not both, except 0_X and 1_X.

(i) Example. Fuzzy discrete space. Let

$\mathbf{F} = I^X$

Then every fuzzy set is open.

$\Rightarrow$ automatically a fuzzy door space (open door space)

(ii) Example. Fuzzy indiscrete space.

Let $\mathbf{F} = \{0_X, 1_X\}$.

In this fuzzy topology, every fuzzy set is closed. $\Rightarrow$ fuzzy door space.

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(iii) Example. Let X = {a, b}.

Define a fuzzy set:

S(a) = 1, S(b) = 0.

i. e. S = {(a, 1), (b, 0)}.

Let **F** = $\{0_X, S, 1_X\}$.

Then for any fuzzy set A, either $A \in$ **F** or $A^c \in$ **F.**

So (X, **F**) is a fuzzy door space.

5.17 Fuzzy Closure of a Set

The closure of a fuzzy set A is defined as the intersection (infimum) of all fuzzy closed sets containing A and is denoted by Cl(A) or $\bar{A}$.

Symbolically,

$Cl(A) = \wedge \{F : F \text{ is fuzzy closed and } A \leq F\}$.

i.e. Cl(A) is the smallest fuzzy closed set that contains A.

5.18 Basic properties

For any fuzzy sets $A, B \subset X$:

(i) Extensively

$A \leq Cl(A)$.

(ii) Idempotence

Cl(Cl((A) = Cl(A).

(iii) Monotonicity

$A \leq B \Rightarrow Cl(A) \leq Cl(B)$.

(iv) Preserves Union

$Cl(A \vee B) = Cl(A) \vee Cl(B)$.

(v) Boundary Cases

$Cl(0_X) = 0_X$, $Cl(1_X) = 1_X$.

5.19 Example. Let {a, b}.

Define a fuzzy topology:

F = $\{0_X, U, 1_X\}$.

Where U = {(a, 1), (b, 0.4)}.

Let A = {(a, 0.6), (b, 0.3)}.

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The fuzzy closed sets are $U^c = \{(a, 0), (b, 0.6)\}$.

$1_X^c = 0_X$

So fuzzy closed sets are:

$\{1_X, U^c, 0_X\}$.

Choose those containing A.

$A \leq F$

Only 1_X satisfies this

So, $Cl(A) = 1_X$.

Therefore, here the closure of A is the whole fuzzy space.

5.20 Fuzzy Interior of a Set

The interior of a fuzzy set A is defined as the union (supremum) of all fuzzy open sets contained in A and denoted by int(A) or Å.

Symbolically

$Int(A) = \vee \{U \in F : U \leq A\}$

Equivalently,

$Int(A) = (Cl(A^c))^c$

5.21 Fundamental properties

For any fuzzy sets A, B, $\subset$ X:

(i) Int(A) is the largest fuzzy open set contained in A.

(ii) Contrastive

$Int(A) \leq A$

(iii) Idempotent

$Int(Int(A)) = Int(A)$

(iv) Monotone

$A \leq B \Rightarrow Int(A) \leq Int(B)$

(v) Preserves intersections

$Int(A \wedge B) = Int(A) \wedge Int(B)$

(vi) Boundary cases:

(a) $Int(0_X) = 0_X$

(b) $Int(1_X) = 1_X$

5.22 Example. Let X = {a, b} and define a fuzzy topology

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$\mathbf{F} = \{0_X, U, 1_X\}$.

Where U = {(a, 0.8), (b, 0.3)}.

Take a fuzzy set

A = {(a, 0.9), (b, 0.4)}.

Find fuzzy open sets contained in A.

We need $U \leq A$

(a) $0_X \leq A$

(b) $U \leq A$

So admissible fuzzy open sets: 0_X and U.

Take their union (Supremum)

Int(A) = U.

5.23 Fuzzy Exterior of a Set

The exterior of a set A, denoted by ext(A), is defined as:

$Ext(A) = Int(A^c)$.

Thus, the fuzzy exterior of A is the largest fuzzy open set completely outside A.

It represents the region that surely does not belong to A.

So,

Exterior = Interior of the complement

Since interior and closure are dual,

$ext(A) = (Cl(A))^c$.

5.24 Basic properties

For fuzzy sets A, B:

(i) Open set

$ext(A) \in \mathbf{F}$

(ii) Disjointness

$ext(A) \cap A = 0_X$

(iii) Monotonicity

$A \leq B \Rightarrow ext(B) \leq ext(A)$

(iv) Boundary Cases

$ext(0_X) = 1_X$, $ext(1_X) = 0_X$

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(v) Relation with interior

$$ext(A) = Int(A^c)$$

5.25 Example. Let X = {a, b} and define a fuzzy topology:

F = $\{0_X, U, 1_X\}$.

Where U = {(a, 0.7), (b, 0.2)}.

Take A = {(a, 0.4), (b, 0.6)}

Then A^c = {(a, 0.6), (b, 0.4)}.

The fuzzy closed sets are U^c = {(a, 0), (b, 0.6)}.

The interior of A^c

Check which fuzzy open sets lie inside A^c :

$0_X \leq A^c$

So, only 0_X qualifies.

Hence $ext(A) = 0_X$.

Fuzzy exterior = biggest fuzzy open set outside the given set.

5.26 Boundary of a fuzzy Set

The boundary of a fuzzy set A, denoted by bd(A) or δ(A), is defined as

$bd(A) = Cl(A) \cap Cl(A^c)$.

5.27 Basic properties

For any fuzzy sets A, B ⊂ X

(i) Symmetry

$$bd(A) = bd\ (A^c).$$

(ii) Non-negativity

$$0_X \leq bd(A).$$

(iii) Boundary of empty or whole space

$$bd(0_X) = 0_X,\ bd(1_X) = 1_X.$$

(iv) Disjointness with interior and exterior

$$bd(A) \wedge Int(A) = 0_X.$$

(v) Decomposition

$$Int(A) \vee bd(A) \vee ext\ (A) = 1_X.$$

5.28 Example. Let X = {a, b} and define a fuzzy topology:

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F = {0_X, U, 1_X}.

Where U = {(a, 0.6), (b, 0.3)}.

Take A = {(a, 0.5), (b, 0.4)}

Then A^c = {(a, 0.5), (b, 0.6)}.

Cl(A) = 1_X, Cl(A^c) = 1_X.

So, every point lies on the fuzzy boundary.

Equivalent Forms:

(i) bd(A) = Cl(A) – Int (A)

(ii) bd(A) = (Int(A) $\vee$ ext(A))^c

where

ext(A) = Int (A^c)

Thus, the fuzzy boundary consists of points that belong neither clearly inside A nor clearly outside A.

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