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THE ISSUE OF THE NUMBER OF "INTEGER TRIANGLES"

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ABSTRACT

This article highlights one of the most pressing problems in the combination of geometry and number theory, the problem of the number of "integer triangles" and some interesting lemmas.

KEYWORDS***: integer, triangle, lemma, theorem, number, side, inequality, set, even, odd, perimeter, formula.***

DISCUSSION

An integer triangle is a triangle whose sides are integers. Let the problem be: Find the number of triangles whose perimeter is 10 and whose sides are integers. It can be easily determined that such triangles may be $(2,4,4)$ and $(3,3,4)$. However, as the perimeter value increases, the problem becomes more complicated and it leads our likelihood of making a mistake in problem solving to increase. Therefore, we present the proof of the following theorem.

Before giving the theorem, let us enter the following definitions.

(1) $\lfloor x \rfloor - x$ The largest integer smaller than or equal to *x.*

$\lceil x \rceil - x$ The smallest integer greater than or equal to *x.*

(2) (i) If x is integer, here $\lfloor x \rfloor = \lceil x \rceil = x$.

(ii) If x is not integer, $\lceil x \rceil = \lfloor x \rfloor + 1 \Rightarrow \lceil -x \rceil + \lfloor x \rfloor = 0$, i.e, $\lceil x+n \rceil = \lceil x \rceil + n$ and $\lfloor x+n \rfloor = \lfloor x \rfloor + n$, here $n \in Z$.

(3) $\tau(x) - x$ is the closest number to the integer. For example, $\tau(2,1) = 2, \tau(2,6) = 3$ and etc.

(This function is defined continuously for $x - \lfloor x \rfloor \neq \frac{1}{2}$).

Now we present the main theorem in the article with full proof.

Theorem: The number of triangles, whose sides are natural numbers and whose perimeter is n that equal to a given number:

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$$T(n)=\begin{cases}\tau\left(\dfrac{n^2}{48}\right) & ,if,n-even\\ \tau\left(\dfrac{(n+3)^2}{48}\right) & ,if,n-odd\end{cases}$$

Proof: Let the sides of a triangle be a,b,c, here we determine $a\geq b\geq c$ without limiting the generality. We obtain the relation $b+c>a\geq b\geq c$ (1) using the triangle inequality, and the fulfillment of this inequality is also a sufficient condition for the existence of a triangle whose sides are equal a,b,c in length. Let's take a collection as follows $A_n=\{(a,b,c)\mid a+b+c=n;b+c>a\geq b\geq c\}$:

In that case, the number of elements in the set A_n is equal to the number of triangles we need, i.e. $|A_n|=T(n)$

We must first limit a: from the relation (1), $b+c>a$ or $n-a>a$, in that case $n>2a$ or $n-1\geq 2a$, i.e. $a\leq\left\lfloor\frac{n-1}{2}\right\rfloor$.

Moreover, according to the relation (1) $n=a+b+c\leq 3a$ or $\left\lceil\frac{n}{3}\right\rceil\leq a$, that is $\left\lceil\frac{n}{3}\right\rceil\leq a\leq\left\lfloor\frac{n-1}{2}\right\rfloor$.

Now let's limit b in relation to a: $n=a+b+c\leq a+2b$, i.e. $\left\lceil\frac{n-a}{2}\right\rceil\leq b\leq a$.

Obviously, there is a natural number $a-\left\lceil\frac{n-a}{2}\right\rceil+1$ between $\left\lceil\frac{n-a}{2}\right\rceil$ and a, so b can be chosen exactly the same way. We can match exactly one c for each (a,b) pair by the formula $c=n-(a+b)$, because the above inequalities are directly derived from the triangular inequality. We find that for each a, there is b, that has $a-\left\lceil\frac{n-a}{2}\right\rceil+1$, so if we find the number of pairs , we can find $|A_n|$.

Thus, it is enough to calculate $|A_n|=\sum_{a=\left\lceil\frac{n}{3}\right\rceil}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(a-\left\lceil\frac{n-a}{2}\right\rceil+1\right)$. Now, to calculate this sum, we give the following auxiliary lemmas with proof.

Lemma: For any natural number n, the following equation holds:

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor=\left\lfloor\frac{n^2}{4}\right\rfloor=\frac{1}{4}\left(n^2-\frac{1-(-1)^n}{2}\right)$$

Proof of the lemma: We prove this in two cases:

Case 1: Let $n=2m$. Here,

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$$\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor=\left\lfloor\frac{1}{2}\right\rfloor+\left\lfloor\frac{2}{2}\right\rfloor+\ldots+\left\lfloor\frac{2m-1}{2}\right\rfloor+m=2(1+2+\ldots+m-1)+m=m(m-1)+m=m^2=\left\lfloor\frac{n^2}{4}\right\rfloor$$

Case 2: Let $n=2m+1$.

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor=\left\lfloor\frac{1}{2}\right\rfloor+\left\lfloor\frac{2}{2}\right\rfloor+\ldots+\left\lfloor\frac{2m}{2}\right\rfloor+\left\lfloor\frac{2m+1}{2}\right\rfloor=2(1+2+\ldots+m)=m(m+1)=n^2-1=\left\lfloor\frac{n^2}{4}\right\rfloor$$

.

Thus, for any $n\in\mathbb{N}$, it holds $\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor=\left\lfloor\frac{n^2}{4}\right\rfloor$.

On the other hand, it is possible to say $\left\lfloor\frac{n^2}{4}\right\rfloor=\frac{n^2}{4}-\frac{1-(-1)^n}{8}$, Because, if n is even, $\left\lfloor\frac{n^2}{4}\right\rfloor=\frac{n^2}{4}-\frac{1-1}{8}=\frac{n^2}{4}$ and if n is odd, $\left\lfloor\frac{n^2}{4}\right\rfloor=\frac{n^2}{4}-\frac{1+1}{8}=\frac{n^2-1}{4}$. In this case, the following result can be obtained from the lemma:

$$\sum_{k=m}^{n}\left\lfloor\frac{k}{2}\right\rfloor=\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor-\sum_{k=1}^{m-1}\left\lfloor\frac{k}{2}\right\rfloor=\frac{1}{4}\left(n^2-\frac{1-(-1)^n}{2}-(m-1)^2+\frac{1-(-1)^{m-1}}{2}\right)=$$
$$\frac{n^2-(m-1)^2}{4}+\frac{(-1)^n+(-1)^m}{8}\quad(2)$$

By the use of the (2) identity, we calculate $\sum_{a=\left\lceil\frac{n}{3}\right\rceil}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(a-\left\lceil\frac{n-a}{2}\right\rceil+1\right)$.

For $LCM(2,3)=6$, n is determined by $(\mathrm{mod}\,6)$.

Case 1. Let $n=6k$, thus n is an even number.

$$\sum_{a=\left\lceil\frac{n}{3}\right\rceil}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(a-\left\lceil\frac{n-a}{2}\right\rceil+1\right)=\sum_{a=2k}^{3k-1}\left(a-\left\lceil 3k-\frac{a}{2}\right\rceil+1\right)=\sum_{a=2k}^{3k-1}\left(a-3k+1-\left\lceil-\frac{a}{2}\right\rceil\right)=(ii)=\sum_{a=2k}^{3k-1}\left(a-3k+1+\left\lfloor\frac{a}{2}\right\rfloor\right)=$$
$$\sum_{a=2k}^{3k-1}(a-3k+1)+\sum_{a=2k}^{3k-1}\left\lfloor\frac{a}{2}\right\rfloor=\frac{3k(3k-1)}{2}-\frac{2k(2k-1)}{2}-3k^2+k+$$
$$\frac{(3k-1)^2-(2k-1)^2}{4}+\frac{(-1)^{3k-1}+(-1)^{2k}}{8}=\frac{5k^2-k}{2}-3k^2+k+\frac{5k^2-2k}{4}+\frac{(-1)^{3k-1}+1}{8}=$$
$$\frac{10k^2-2k-12k^2+4k+5k^2-2k}{4}+\frac{(-1)^{3k-1}+1}{8}=\frac{3k^2}{4}+\frac{(-1)^{3k-1}+1}{8}=\frac{n^2}{48}+\frac{(-1)^{\frac{n-2}{2}}+1}{8}=\tau\left(\frac{n^2}{48}\right)$$

Case 2. Let $n=6k+1$, thus n is an odd number.

$$\sum_{a=\left\lceil \frac{n}{3} \right\rceil}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(a - \left\lceil \frac{n-a}{2} \right\rceil + 1\right) = \sum_{a=2k}^{3k} \left(a - 3k - \left\lceil \frac{1-a}{2} \right\rceil + 1\right) = |(ii)| = \sum_{a=2k+1}^{3k} \left(a - 3k + 1 + \left\lfloor \frac{a-1}{2} \right\rfloor\right) =$$

$$\sum_{a=2k+1}^{3k} (a-3k+1) + \sum_{t=2k}^{3k} \left\lfloor \frac{t}{2} \right\rfloor = \frac{3k(3k+1)}{2} - \frac{2k(2k+1)}{2} - 3k^2 + k + \frac{(3k-1)^2 - (2k-1)^2}{4} +$$

$$\frac{(-1)^{3k-1} + (-1)^{2k}}{8} = \frac{5k^2 + k}{2} - 3k^2 + k + \frac{5k^2 - 2k}{4} + \frac{(-1)^{3k-1} + 1}{8} =$$

$$\frac{10k^2 + 2k - 12k^2 + 4k + 5k^2 - 2k}{4} + \frac{(-1)^{3k-1} + 1}{8} = \frac{3k^2 + 4k}{4} + \frac{(-1)^{3k-1} + 1}{8} =$$

$$\frac{(n+3)^2}{48} - \frac{1}{3} + \frac{(-1)^{\frac{n-3}{2}} + 1}{8} = \tau\left(\frac{(n+3)^2}{48}\right).$$

So, in this case the theorem is also correct. The rest of the cases prove the same.

Let us examine the problem we have first posed by this theorem. The problem was: Find the number of triangles whose perimeter is 10 and whose sides are integers. So considering that 10 is even, we have equality $\tau\left(\frac{10^2}{48}\right) = \tau(2{,}08...) = 2$. The answer to our question above was 2. Now, as each student enlarges the perimeter of the triangle and tries to count the number of triangles using this theorem and some other method, he or she will better understand the significance of the theorem.

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