SPECTRAL-GRID METHOD FOR SOLVING EVOLUTION PROBLEMS WITH HIGH GRADIENTS

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ABSTRACT
For numerical modeling, the Burgers equation describing the motion of nonlinear wave systems is more and more widely used numerical methods. At the same time, their application to the solution of evolutionary problems with large gradients, described by non-stationary partial differential equations, is subject to serious difficulties. In this paper, the spectral-grid method is used to numerically simulate Burgers equation. In the spectral-grid method, the interval of integration over the spatial variable is divided into a grid, in the grid elements the approximate solution is approximated with the help of a linear combination of a different number of series in Chebyshev polynomials of the first kind.
KEYWORDS: *mathematical modeling, nonlinear wave systems, evolution problems, spectral-grid method, interval of integration, approximation grid, Chebyshev polynomials of the first kind, algebraic system, approximate solution, efficiency.*

INTRODUCTION

Numerical solutions of the initial-boundary value problems of hydro-aerodynamics with large gradients encounters serious difficulties due to the lack of sufficient spatial resolution in regions of strong inhomogeneity. These difficulties in many cases are overcome by using spectral methods and their modifications, which have the property of high-precision spatial approximation with an increase in the number of basis functions. In applied calculations, however, it is not always possible to increase the number of basis functions to the required value, since the growth of the order of matrices in the resulting algebraic system is significantly limited by computer resources.

In this work, to overcome these difficulties, the spectral-grid method is used [1-7]. Depending on the type of initial data or the expected form of the solution, a grid is introduced in the integration interval. At the internal nodes of the grid, the requirement is imposed on the continuity of the solution and its derivatives up to order $m-1$, where m is the order of the differential equation. At the boundary grid nodes, the corresponding boundary conditions for the problem under consideration are set. An approximate solution on grid elements is represented in the form of finite series in Chebyshev polynomials of the first kind. The resulting system of equations using linear non-degenerate transformations is reduced to two autonomous systems: a linear system of algebraic equations and a system (in the general case of nonlinear) ordinary differential equations. To solve the first system, standard methods are used, and to solve the second, an explicit algorithm developed in [12] is used.

Therefore, the use of the spectral-grid method makes it possible, firstly, to distribute the Chebyshev polynomials over the elements, taking into account the behavior of the solution gradient and, secondly, to lead to a significant decrease in the order of the matrices in the arising algebraic system. In this method, for a given number of grid elements N, to achieve the required accuracy of calculations, it is necessary to correctly position the grid nodes and select the number of polynomials p_j on the grid elements. These questions are closely related, since by bringing the grid nodes closer together, one can reduce the number of polynomials on the elements and vice versa. In practical calculations, it is more convenient to choose a uniform mesh, setting different numbers of polynomials p_j on each mesh element. Then the number of required polynomials depends

on the relative magnitude of the gradients of the solution at one or another element. Solution gradients can often be estimated from asymptotic analysis. In problems with large gradients, it is known [13] that near the wall — in the so-called critical layer — the behavior of the solution is determined by a rapid change in viscous solutions; far from the wall, the perturbations slowly decay.

MAIN PART

One of the important and difficult mathematical problems is related to the description of the behavior of low-amplitude waves that experience weak dissipation over long time intervals [1]. These restrictions are not as special as they might seem at first glance. Since, as follows from observations, waves can indeed exist for a long time outside the sources, the limitations associated with the assumption of low dissipation and large time intervals are quite natural. In gas dynamics, there is weak dissipation, characterized by a dimensionless parameter Re^{-1}, where Re is the Reynolds number. In this case, the magnitude of the amplitude of wave ε is small, but finite. In cases of greatest interest, the corresponding nonlinear equation should be considered at time intervals of the order of ε^{-1}.

The studied process is described by the Burgers equation. We consider the following initial-boundary value problem

$$\frac{\partial u}{\partial t} - \frac{1}{\mu}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0, \quad a < x < b, \tag{1}$$

$$u(a,t) = 0,$$
$$u(b,t) = 0, \tag{2}$$
$$u(x,0) = u_0(x). \tag{3}$$

Let's break the integration interval $[a,b]$ into M different elements:
$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{i-1}, x_i], \dots, [x_{M-1}, x_M],$$

where $x_0 = a$, $x_M = b$. To represent the approximate solution in the form of series in Chebyshev polynomials, each element $[x_{i-1}, x_i]$ of the integration interval $[a,b]$ is mapped to the interval $[-1,1]$ using the following replacement of the independent variable

$$x_i = \frac{m_i}{2} + \frac{l_i}{2}y, \tag{4}$$

where $m_i = x_i + x_{i-1}$, and $l_i = x_i - x_{i-1}$ is the length of the i th mesh element and $y \in [-1,1]$. After this transformation, problem (1) - (3) takes the form:

$$\frac{\partial u_i}{\partial t} = \frac{1}{\mu}\left(\frac{2}{l_i}\right)^2 \frac{\partial^2 u_i}{\partial y^2} - \frac{2}{l_i}u_i\frac{\partial u_i}{\partial y}, \quad i = 1,2,\dots,M \tag{5}$$

$$u_i(1) = u_{i+1}(-1), \quad i = 1,2,\dots,M-1, \tag{6}$$

$$\frac{1}{l_i}\frac{\partial u_i}{\partial y}(1) = \frac{1}{l_{i+1}}\frac{\partial u_{i+1}}{\partial y}, \quad i = 1,2,\dots,M-1, \tag{7}$$

$$u_{01}(-1) = u_{0N}(1) = 0, \tag{8}$$

$$u_{0i}(y,0) = u_0\left(\frac{m_i}{2} + \frac{l_i}{2}y,0\right), \quad i = 1,2,\dots,M, \tag{9}$$

where equations (6) - (7) are the requirements for the continuity of the approximate solution and its first derivative at the internal nodes of the grid, equation (8) is the form of the initial data (the initial data for the subsequent position are of no fundamental importance and therefore are not considered).

An approximate solution of equations (5) - (8) will be sought in the form of series in Chebyshev polynomials of the first kind $T_n(y)$ [8-11]:

$$u_j(y) = \sum_{n=0}^{N} a_n^j T_n(y),$$

$$T_n(y) = \cos(n \cdot \arccos y) \tag{10}$$

where N is the number of polynomials used for approximation at the j st element. We choose to each of elements M, $N+1$ at discrete points $y_l = \cos(\pi l/N), l = 0,1,..., N$ and write system (5) - (8), and these points. Expansion coefficients a_m^j for function $u_j(y_e)$ are determined by the inverse transformation [3]:

$$a_m^j = \frac{2}{NC_m} \sum_{l=0}^{N} u_j(y_l) T_m(y_l)$$
$$m = 0,1,...,N, \quad j = 1,2,...,M \tag{11}$$
$$C_0 = C_N = 2, C_m = 1, \text{ at } m \neq 0, N .$$

For convenience of further presentation, we write formulas (10) and (11) in matrix form:

$$v = Ta, \tag{12}$$
$$a = T * v, \tag{13}$$

where $a = \{a_0^1, a_1^1,..., a_N^1, a_0^2, a_1^2,..., a_N^2,..., a_0^M, a_1^M,..., a_N^M\}$, is a vector (see (13)), T and $T*$ are block-diagonal matrices of dimension $((N+1)M) \times ((N+1)M)$:

The first and second spatial derivatives can be represented in a similar form at the same discrete points y_l:

$$\frac{\partial v}{\partial y} = Tb, \tag{14}$$

$$\frac{\partial^2 v}{\partial y^2} = Td, \tag{15}$$

moreover, the components of vectors b and d are determined through the components of vector a [1]

$$c_m b_m^j = 2 \sum_{\substack{P=m+1 \\ P \equiv m (\text{mod } 2)}}^{N} P a_P^j, \quad \begin{matrix} m \geq 0 \\ j = 1,...,M \end{matrix} \tag{16}$$

$$c_m d_m^j = 2 \sum_{\substack{P=m+2 \\ P \equiv m (\text{mod } 2)}}^{N} P(P^2 - n^2) a_P^j, \quad \begin{matrix} m \geq 0 \\ j = 1,...,M \end{matrix} \tag{17}$$

Writing $a \equiv b (\text{mod } 2)$ means that $a - b$ is divisible by 2. Formulas (16) and (17) can also be written in matrix form:

$$b = Ra, \tag{18}$$
$$d = Pa. \tag{19}$$

Substituting (18) and (19) into (14) and (15), respectively, and taking into account (18), we arrive at a pseudospectral approximation of the spatial derivatives:

$$\frac{\partial v}{\partial y} = \hat{B}v, \tag{20}$$

$$\frac{\partial^2 v}{\partial y^2} = \hat{A}v, \tag{21}$$

where $\hat{A}$ and $\hat{B}$ denote the following matrix products of dimension $((N+1)M) \times ((N+1)M)$:

$$\hat{A} = TPT*, \quad \hat{B} = TRT*. \tag{22}$$

We also introduce matrices $\tilde{A}$ and $\tilde{B}$:

$$\tilde{A} = \frac{1}{\mu} L^2 \hat{A}, \quad \tilde{B} = L\tilde{B}, \tag{23}$$

where L is a diagonal matrix:
where v is a vector of length $(N+1)M$ with components:

$$v \equiv \{u_1(y_0)...u_1(y_N), u_2(y_0)...u_2(y_N),$$
$$u_3(y_0)...,u_M(y_0)...u_M(y_N),\}$$

(24)

The steps for solving the problem are described in [15].
Thus, the sequence of calculations is as follows:

1) the region of integration is divided into a certain number of elements M ;

2) a one-step algorithm (for example, the Runge-Kutta method) finds a solution on the first two time layers: $t = \tau, t = 2\tau$;

3) a number of nondegenerate transformations of matrix H are carried out in order to weaken the stability condition: $\dfrac{dr}{dt} = Hr - f$;

4) according to formula

$$r(t+\tau) = r(t) + RQr(t) - Rf(t),$$

$$R\varphi(t) = \frac{\tau}{12}[23\varphi(t) - 16\varphi(t-\tau) + 5\varphi(t-2\tau)],$$

$$Q = \frac{12}{\tau} \frac{(e^{H\tau} - E)}{(23E - 16e^{-H\tau} + 5e^{-2H\tau})}$$

the transition to a new time layer $t = \tau$ is carried out;

5) the components of the vector v at the boundary points of the elements are found from the solution of the algebraic system $w = Gv$.

RESULTS AND DISCUSSION

The constructed algorithm was applied to calculations of one-dimensional initial-boundary value problems for the Burgers equations.

A similar effect can be observed in the case of nonlinear evolutionary equations. Below we consider the initial-boundary value problem for the Burgers equation:

$$\frac{\partial u}{\partial t} - \frac{1}{\mu}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0,$$

(25)

$$u(x,0) = -\sin \pi x,$$

$$u(\pm 1, t) = 0.$$

(26)

The analytical solution of problem (25) - (26) has the form of the ratio of series of Bessel functions [16]. For large numbers $\mu(\mu \geq 10^2)$, it is difficult to compare the numerical solution of problem (25) - (26) with the analytical one due to the slow convergence of the series in the analytical solution [12]. Therefore, the calculation accuracy was estimated by the order of magnitude of the last coefficients a_n^j in expansion (10) (the order of error does not exceed the indicated value).

Specific calculations were carried out for the values of the parameters: $\mu = \pi \cdot 10^2, n = 64, \tau = 10^{-2}/6\pi$. The results are shown in Figs. 1 and 2. In both figures, curves 1, 2, 3, 4 show the behavior of the numerical solution at time $t = 0., 0.2, 0.5, 1.0$, and edge 5 in Fig. 2 corresponds to time $t = 2.0$. It can be seen in Fig. 1 $(M = 1)$ that with an increase in the first derivative at point $x = 0$, the numerical solution begins to oscillate, and at $t = 1.0$ the amplitude of the oscillations increases so much that not a single correct sign remains in the solution. In the case of dividing the region of integration into 2 elements: $[-1,0], [0,1]$ (Fig. 2), the amplitude of oscillations at $t = 0.5$ is much less - the solution has 2 correct signs. When the $t > 0.5$ oscillations are smoothed out, the accuracy of the solution increases again - by time $t = 2.0$, for example, it reaches $\varepsilon \sim 10^{-4}$.
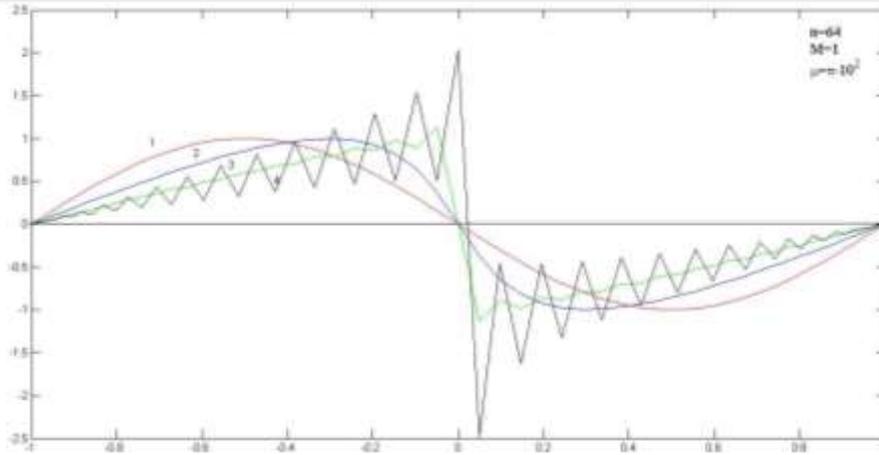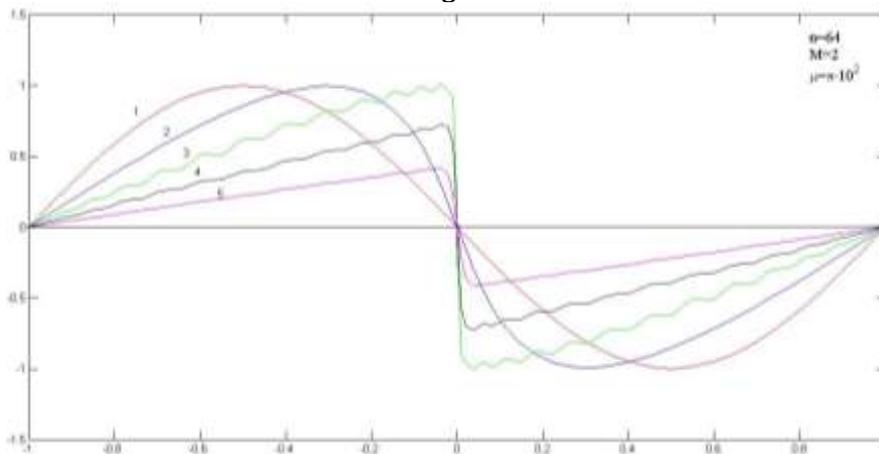
Fig.1

Fig.2

Figure 3 shows the evolution of the numerical solution for the case when the integration interval is divided into a larger number of elements (M = 8).

It can be seen that for all x, except for a narrow region near $x = 0$, where the amplitude of the oscillations is negligible, there are no oscillations for all values of t.

Thus, the calculation results show that by dividing the integration interval of ka elements, the accuracy of the numerical solution in the region of large gradients can be significantly increased.

It is also of some interest to compare the described method with the method described in [12]. As in this work, in [12], the spectral-grid approximation of the spatial derivatives was used, and on the elements the solution was approximated by series in Chebyshev polynomials. Table 1 shows the value $\max\left|\dfrac{\partial u}{\partial x}\right|$, calculated by both methods, as well as the exact value of this value, found from the analytical solution.

Here $t_{\max}$ is the time point at which

Table 1.
Calculation results (here $\mu = 100\pi$)

Method	Interval	$\left\|\dfrac{\partial u}{\partial x}\right\|_{\max}$	$\pi t_{\max}$	$N \times M$	$\pi\tau$
1. Spectral grid method (this work)	[-1,1]	152.4	1.6038	16×4	$10^{-\frac{2}{6}}$
2. Spectral elemental (Patera. A.T [9])	[-1,1]	152.0	1.6033	16×4	$10^{-\frac{2}{6}}$
3. Analytical solution		152.0	1.6037		

reaches a maximum. The values of all parameters in both cases are the same. It is seen that in both cases the accuracy of the calculations is of the same order. The advantage of the proposed method, however, lies in its greater generality, since, unlike [14], it does not contain any assumptions about the form of the solution, and therefore does not require the introduction of additional parameters determined in the calculation process.
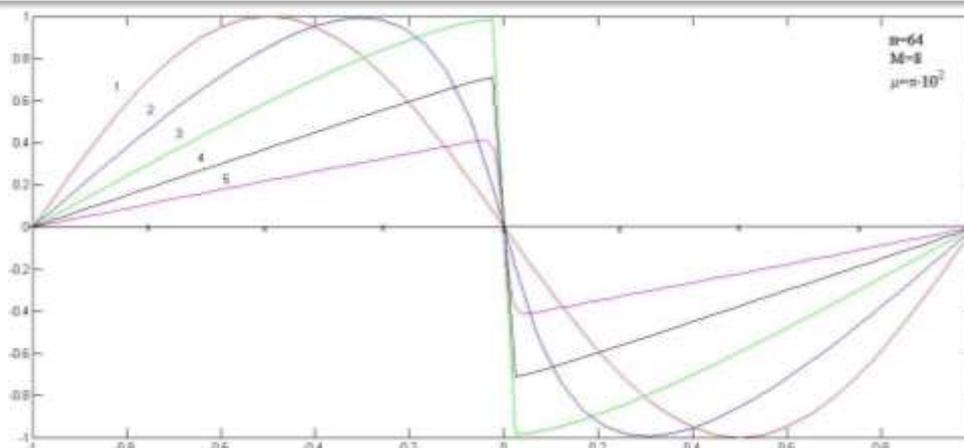
Fig. 3

CONCLUSIONS

1. A spectral-grid explicit method for solving evolutionary problems with large gradients is constructed. Depending on the location of the regions of inhomogeneity, the integration interval is divided into a finite number of elements. On each of the elements, spectral approximation by finite series in basis functions is used.

2. Concrete calculations for the one-dimensional Burgers equation have been carried out, and a comparison has been made with the one-element pseudospectral method.

3. It is shown that the use of the spectral-grid approximation makes it possible to significantly increase the accuracy of calculations without increasing the total number of basis functions.

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