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CHARACTERISTICS OF SMOOTH MAPPING AND SINGULAR POINTS

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ABSTRACT

This paper is devoted to the study of smooth mappings and it has been shown its mapping of surface into 3-dimensional spaces, as well as Thom-Boardman singularities. Certain other results have also been proved.

In mathematics, singularity theory studies spaces that are almost manifolds, but not quite. A string can serve as an example of a one-dimensional manifold, if one neglects its thickness. A singularity can be made by balling it up, dropping it on the floor, and flattening it. In some places the flat string will cross itself in an approximate "X" shape. The points on the floor where it does this are one kind of singularity, the double point: one bit of the floor corresponds to more than one bit of string. Perhaps the string will also touch itself without crossing, like an underlined "U". This is another kind of singularity. Unlike the double point, it is not stable, in the sense that a small push will lift the bottom of the "U" away from the "underline".

Vladimir Arnold describes the main goal of singularity theory as describing how objects depend on parameters, particularly in cases where the properties undergo sudden change under a small variation of the parameters. These situations are called perestroika bifurcations or catastrophes. Classifying the types of changes and characterizing sets of parameters which give rise to these changes are some of the main mathematical goals.

A simple example might be the outline of a smooth object like a kidney bean. From some angles the outline is a smooth curve but as the object is rotated, the outline will first form a sharp corner and then a self-intersection with cusps. Singularities can occur in a wide range of mathematical objects, from matrices depending on parameters to wave fronts.

1-Preliminaries

Singularities arise naturally in a huge number of different areas of mathematics and science. As a consequence Singularity Theory lies at the cross roads of the paths connecting the most important areas of applications of mathematics with its most abstract parts. For example, it connects the investigation of optical caustics with simple Lie algebras and regular polyhedral theory, while also relating hyperbolic PDE wave fronts to knot theory and the theory of the shape of solids to commutative algebra.

The main goal in most problems of singularity theory is to understand the dependence of some objects of analysis and geometry, or physics, or from some other science on parameters. For generic points in the parameter space their exact values influence only the quantitative aspects of the phenomena, their qualitative, topological features remaining stable under small changes of parameter values.

However, for certain exceptional values of the parameters these qualitative features may suddenly change under a small variation of the parameter. This change is called a perestroika, bifurcation or catastrophe in different branches of the sciences. A typical example is that of Morse surgery, describing the perestroika of the level variety of a function as the function crosses through a critical value. (This has an important complex counterpart - the Picard-Lefschetz theory concerning the branching of integrals.) Other familiar examples include caustics and outlines or profiles of surfaces obtained from viewing or projecting from a point, or in a given direction.

In spite of its fundamental character, and the central position it now occupies in mathematics, singularity theory is a surprisingly young subject. So, for example, one can consider the singularities arising from the

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orthogonal projections a generic surface in 3-space, a problem of surely classical interest. Their classification was completed as recently as 1979.

In one sense singularity theory can be viewed as the modern equivalent of the differential calculus, and this explains its central position and wide applicability. In its current form the subject started with the fundamental discoveries of Whitney (1955), Thom (1958), Mather (1970), Brieskorn (1971). Substantial results and exciting new developments within the subject have continued to flow in the intervening years, while the theory has embodied more and more applications.

This program will bring together experts within the field and those from adjacent areas where singularity theory has existing or potential application. Applications of particular interest include those to wave propagation, dynamical systems, quantum field theory, and differential and algebraic geometry, but these should not be deemed prescriptive. It is the program's aim both to foster exciting new developments within singularity theory, and also to build bridges to other subjects where its tools and philosophy will prove useful.

2-How Singularities may arise

In singularity theory the general phenomenon of points and sets of singularities is studied, as part of the concept that manifolds (spaces without singularities) may acquire special, singular points by a number of routes. Projection is one way, very obvious in visual terms when three-dimensional objects are projected into two dimensions (for example in one of our eyes); in looking at classical statuary the folds of drapery are amongst the most obvious features. Singularities of this kind include caustics, very familiar as the light patterns at the bottom of a swimming pool.

Other ways in which singularities occur is by degeneration of manifold structure. The presence of symmetry can be good cause to consider orbifolds, which are manifolds that have acquired "corners" in a process of folding up, resembling the creasing of a table napkin.

3 - Arnold's view

While Thom was an eminent mathematician, the subsequent fashionable nature of elementary catastrophe theory as propagated by Christopher Zeeman caused a reaction, in particular on the part of Vladimir Arnold.[2] He may have been largely responsible for applying the term singularity theory to the area including the input from algebraic geometry, as well as that flowing from the work of Whitney, Thom and other authors. He wrote in terms making clear his distaste for the too-publicized emphasis on a small part of the territory.

The foundational work on smooth singularities is formulated as the construction of equivalence relations on singular points, and germs. Technically this involves group actions of Lie groups on spaces of jets; in less abstract terms Taylor series are examined up to change of variable, pinning down singularities with enough derivatives. Applications, according to Arnold, are to be seen in simplistic geometry, as the geometric form of classical mechanics.

4-Duality

An important reason why singularities cause problems in mathematics is that, with a failure of manifold structure, the invocation of Poincare duality is also disallowed. A major advance was the introduction of intersection chorology, which arose initially from attempts to restore duality by use of strata. Numerous connections and applications stemmed from the original idea, for example the concept of perverse sheaf in homological algebra.

5-Other Possible Meanings

The theory mentioned above does not directly relate to the concept of mathematical singularity as a value at which a function is not defined. For that, see for example isolated singularity, essential singularity, removable singularity. The monodromy theory of differential equations, in the complex domain, around singularities, does however come into relation with the geometric theory. Roughly speaking, *monodromy* studies the way a covering map can degenerate, while *singularity theory* studies the way a *manifold* can degenerate; and these fields are linked definitions.

6-Cone-like singularities

A manifold with singularities of Baas-Sullivan type is a topological space $\bar{A}$that looks like a manifold outside of a compact 'singularity set', while the singularity set has a neighborhood that looks like the product of manifold

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and a cone. Here is a precise definition. Let $P1$ be a closed manifold. A manifold with a $P1$-singularity is a space of the form

$$\bar{A}=A\cup_{A(1)\times P1}A(1)\times CP(1)$$

$$\partial A=A(1)\times P_1$$

Here, A is a manifold with boundary $A(1)$.
More complex singularities occur if, instead of taking a cone over only one manifold $P1$, we allow a collection $\{P1,...,Pk\}$ of several closed manifolds. In this case, we define a a manifold with a $\{P1,...,Pk\}$-singularity to be a (second-countable and Hausdorff) topological space $\bar{A}$ locally homeomorphic to one of the spaces

$$\mathrm{R}^n,\ \mathrm{R}^{n1}\times CP_1,\ \mathrm{R}^{n2}\times CP_1\times CP_2,....$$

An alternative approach to manifolds with singularities would be to remove the singular set and to define an equivalence relation on the remaining manifold that 'remembers' the singularities.

7- Σ-manifolds

An alternative definition can be given. Let $(P1,...,Pk)$ be a (possibly empty) collection of closed manifolds and denote by $P0$ the set containing only one point. Then define $\Sigma k:=(P0,P1,...,Pk)$. For a subset $I=\{i1,...,iq\}\subset\{0,...,k\}$ define $P^I:=Pi1\times...\times Piq$.
A manifold M is a Σk-Manifold if there is given

1. a partition $\partial M=\partial_0 M\cup...\cup\partial_k M$, such that $\partial_I M:=\partial i_1\cap...\cap\partial iq\ M$ is a manifold for each $I=\{i_1,...,iq\}\subset\{0,...,k\}$, and such that

$$\partial(\partial_I M)=\cup_{j\notin I}\partial_j M\cap\partial_I M$$

.

1. for each $I\subset\{0,...,k\}$ a manifold $\beta_I M$ and a diffeomorphism
2.

$$\phi_I:\partial_I M\to\beta_I M\times P^I$$

such that if $J\subset I$ and $\imath:\partial_I M\to\partial_J M$ is the inclusion, then the composition

$$\phi_J\circ\imath\circ\phi^{-1}I:\beta_I M\times P^I\to\beta_J M\times P^J$$

restricts to the identity on the factor P^J in P^I. The diffeomorphisms ϕ_I are called product structures.
On a Σk-manifold M, there is a canonical equivalence relation $\sim$: two points $x,y\in M$ are defined to be equivalent if there is an $I\subset\{0,...,k\}$ such that $x,y\in\partial_I M$ and $pr\circ\phi_I(x)=pr\circ\phi_I(y)$, where $pr:\beta_I M\times P^I\to\beta_I M$ is the projection. Now we can give a general definition: a manifold with a Σk-singularity is a topological space $\bar{A}$ of the form

$$\bar{A}=A/\sim$$

for a Σk-manifold A.
The spaces defined above as manifolds with a $(P1,...,Pk)$-singularity are contained in this new definition. Given manifolds $P1,...,Pk$, set $\Sigma k=(P0,P1,...,Pk)$. Removing a neighborhood of the cone-tips in a manifold with $(P1,...,Pk)$-singularity $\bar{A}$gives a Σk-manifold M. Now the collapsing of the equivalence relation in M corresponds to the re-attachment of the cone-ends.
When dealing with manifolds with singularities it is convenient to work with the underlying Σ-manifold and make sure that all operations one performs on them are compatible with the equivalence relation.

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8 -Singularities of differentiable mappings:

A branch of mathematical analysis and differential geometry, in which those properties of mappings are studied which are preserved when the coordinates in the image and pre-image of the mapping are changed (or when changes are made which preserve certain supplementary structures); a general approach is proposed to the solution of various problems on degeneration of mappings, functions, vector fields, etc.; a classification is given of the most commonly encountered degenerations, and their normal forms, as well as algorithms which reduce to the normal forms, are determined.

A point of the domain of definition of a differentiable mapping (i.e. a mapping of class C^k, see Differentiable manifold) is said to be regular if the Jacobi matrix has maximum rank at this point, and critical in the opposite case.

The classical implicit function theorem describes the structure of a mapping in a neighborhood of a regular point; in a neighborhood of this point and in a neighborhood of its image, there exist coordinates in which the mapping is linear.

In many cases it is not sufficient to confine the area of study simply to regular points; it is therefore natural to consider the following questions:

(a) the description of a mapping in a neighborhood of a critical point;

(b) the description of the structure of the set of critical points.

For an arbitrary mapping there are no answers to a) and b), for two reasons: In attempting to deal with all mappings, there is no chance of obtaining explicit results (for example, the set of critical points can locally be an arbitrary closed set), and for practical applications it is sufficient to know the answers for only a large set of mappings.

The questions (a) and (b) and many others in the theory of singularities are studied along the following lines:

i. a set of "untypical" and "pathological" mappings is excluded from consideration;
ii. a criterion of "typicality" of a mapping is determined;
iii. it is ascertained that every mapping can be approximated by "typical" mappings;
iv. the "typical" mappings are studied.

The choice of the set of typical mappings depends on the problem to be solved and is not unique: the fewer the mappings that are typical, the easier they are to study, although 2) and 3) require that the set of typical mappings is sufficiently broad and sufficiently constructively defined.

9-Singularities of Smooth Maps

A singular point of a smooth mapping : M → N of manifolds is a point at which the rank of f is less than the minimum of dimensions of M and N.

Singularities of smooth mappings have a nice classification, with respect to which for almost any smooth mapping f, the set of singular points of any type $\mathcal{J}$ forms a smooth sub manifold $S_{\mathcal{J}}(f) \subset$M. We study those topological properties of the set

$S_{\mathcal{J}}(f)$ that does not change under homotopy of f.

One of the first questions that arises in the singularity theory asks whether a singularity type $\mathcal{J}$ is in essential for a mapping f; in other words, does there exist a homotopy of f eliminating all the $\mathcal{J}$-singular points? The primary obstruction is defined as the chorology class $[\overline{S_{\mathcal{J}}(f)}] \in H^*(M;\mathbb{Z}_2)$ dual to the closure of $S_{\mathcal{J}}(f)$ .Remarkably, the class $[\overline{S_{\mathcal{J}}(f)}]$ is a polynomial, called Thom polynomial. In Stiefel-Whitney classes of the tangent bundle TM and the induced bundle f*TN.

The Thom polynomial turns out not to be a complete obstruction; O. Saeki constructed an example of a mapping from a 4-manifold into a 3-manifold where the chorology obstruction corresponding to certain singularities, cusps, is trivial though a homotopy to a general position mapping without cusp singular points does not exist.

We consider smooth mappings of 4-manifolds into 3-manifolds, determine the secondary obstruction, prove its completeness and express it in terms of the chorology ring of the source manifold.

Definition : A general position mapping of a 4-manifold into 3-manifold without cusp singular points is called a fold mapping.

Theorem: For a closed oriented 4-manifold M^4, the following conditions are equivalent:

i. M^4 admits a fold mapping into $\mathbb{R}^3$;
ii. for every orient able 3-manifold N^3, every homotopy class of mappings of M^4 into N^3 contains a fold mapping;
iii. there exists a chorology class $\in 2\, H^2(M^4; \mathbb{Z})$ such that $x \cup x$ is the first Pontrjagin class of M^4.

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For a simply connected manifold M^4, we show that M^4 admits no fold mappings into N3 if and only if M4 is homotopy equivalent to CP^2 or $CP^2\#CP^2$.

10-Regular Points of Smooth Mappings

Given a smooth mapping f of a manifold M of dimension m into a smooth manifold N of dimension n, the differential $df(x)$ of the mapping f at a point x of M is a linear map from the tangent space T_xM of M at x to the tangent space$T_{f(x)}$N of N at $f(x)$,

$$df(x) : T_xM \rightarrow T_{f(x)}N.$$

We say that $x \in M$ is a regular point of the mapping f if the rank $rk_x(f)$ of the differential $df(x)$ is exactly max(m, n). Otherwise we say that the point x is
a singular point of the mapping f.
We observe that the set of regular points forms an open sub manifold of the source manifold. Indeed, if a homomorphism h of vector spaces sends a set $\{e_i\}$ of independent vectors into a set of independent vectors, then every homomorphism sufficiently close to h also sends the vectors $\{e_i\}$ into independent ones.
Consequently, if f is a smooth mapping and x is a point of the source manifold, then the rank of the differential $df(x)$ at x is not greater than the rank of the differential $df(y)$at any point y sufficiently close to x. In particular, in a small neighborhood of a regular point, the mapping f has no singular points.
The regular points of a mapping have a simple description. In the case of a positive co dimension, $n - m > 0$, the regular points are precisely the points in a neighborhood of which the mapping f is an embedding. If the mapping f is of a non-positive co dimension, i.e., $n - m \leq 0$, then the regular points are the points of the source manifold in a neighborhood of which the mapping f is a submersion.

11-Singular Points of Smooth Mappings

We study singularities of smooth mappings up to an equivalence relation.
Definition. Given two mappings $f_i : M_i \rightarrow N_i$, i = 1, 2, we say that the points
$x_1 \in M_1$ and $x_2 \in M_2$ are of the same singularity type with respect to the right left equivalence if there are neighborhoods U_i containing x_i, neighborhoods V_i containing $f_i(x_i)$ and diffeomorphisms

$$g : U_1 \rightarrow U_2,\ h : V_1 \rightarrow V_2$$

that fit into the commutative diagram

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_2 \arrow[d] \\ V_1 \arrow[r] & V_2 \end{tikzcd}$$

where the mappings $f_i|U_i$ are the restrictions of the mappings f_i to U_i.
It is convenient to describe a right-left singularity type, say τ by choosing a normal form, i.e., a mapping $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$with singularity τ at the origin. Once the normal form is chosen, we say that a mapping $f: M \rightarrow N$ has τ singularity at a point $x \in M$ if in some local coordinate neighborhoods of x in M and f(x) in N ,the mapping f has the form g.
In the two subsequent sections we will consider examples of singularities in the cases of mappings of manifolds of small dimensions.

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12-Mappings of Surfaces into 3-dimensional Spaces:

One of the singularities of mappings from a surface into a 3-manifold is theWhitney umbrella.
In a neighborhood of a Whitney umbrella, in some coordinates, the mapping f has the form

$$f(u, v) = (uv, u, v^2).$$

Theorem: 1.1 (Whitney) Every mapping of a surface into a 3-manifold can be approximated by a mapping with singularities of only Whitney umbrella type.
The set of Whitney umbrellas is a discrete set. In particular, a mapping of a closed surface may have only finitely many Whitney umbrellas.
In fact, the number of Whitney umbrellas is even. To prove this, we describe the Whitney umbrellas as the end points of self-intersection curves. If the source surface is closed, then each connected component of self-intersection points is either a circle which has no end points or a closed interval which has two end points. Thus the number of Whitney umbrellas is twice the number of closed intervals of self-intersection points.
If we consider mappings up to homotopy, then the Whitney umbrellas are no longer essential; every mapping of a surface into a 3-manifold is homotopic to an immersion. This follows from the Smale-Hirsch h-principle for immersions, which we will discuss in later sections.

13-Mappings Between Surfaces

Singularities of mappings between surfaces were studied by Whitney who proved that every continuous mapping of surfaces can be approximated by a mapping with only regular points, fold singular points, and cusp singular points.
A regular point of a mapping f, as it has been defined, is a point in a neighborhood of which the mapping f is a diffeomorphism.
The fold and cusp singular types are defined by normal forms. We say that a singular point p is of the fold type or the cusp type if in some neighborhoods of p and f(p) there are coordinates in which the mapping f has the form

$$f(x, y) = (x, y^2) \text{ or } f(x, y) = (x^3 + xy, y)$$

respectively As it follows from the normal forms of singularities, the set of singular points S of a mapping f with only fold and cusp singular points forms a smooth curve in the source manifold. The set of cusp points is discrete, while the set of fold points is the 1-dimensional complement to the cusp points in S.
We note that the rank of the differential of the mapping f is 1 both at a point of the fold type and at a point of the cusp type. To distinguish a fold singular point from a cusp singular point, Whitney considered the restriction of the mapping f to the smooth curve of singular points S and observed that the cusp points of f are exactly the singular points of $f \mid S$.
The cusp singular points are essential even if we consider mappings up to homotopy. For example, the projective plane $\mathbb{R}P^2$ does not admit a mapping into$\mathbb{R}^2$ with only fold singular points .
Let us sketch a proof that motivates the notion “homology obstruction. ” We note that any two mappings into $\mathbb{R}^2$ are homotopic. Thus to prove the claim it suffices to construct a mapping $\mathbb{R}P^2 \rightarrow \mathbb{R}^2$ with fold and cusp singularities and then to show that the cusp singular points can not be eliminated by homotopy.

14-Types of Singularities

The right-left equivalence relation on singularities of smooth mappings is so fine that the number of different singularity types of a mapping is infinite in general. Besides, the behavior of the set of points of a right-left equivalence class under homotopy of a mapping has no simple description.
To overcome the difficulties arising here one may consider a coarser relation in which a class of equivalence is a union of some, perhaps infinitely many, right-left equivalence classes of singularities. One of such relations playing a special role in singularity theory is the Thom-Boardman classification.
Every continuous mapping of smooth manifolds admits an approximation by a mapping f with singularities of only finitely many different Thom-Boardman classes. Furthermore, the set of singular points of f of each Thom-Boardman class is a sub manifold of the source manifold.

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14.1- Naive Definition

Let TM and TN denote the tangent bundles of smooth manifolds M and N respectively and df the differential of a smooth mapping $f : M \to N$. The set $S_i = S_i(f)$ is defined as the set of points x in M at which the kernel rank of f is $kr_x f = i$. Suppose that $\dim M = m \geq n = \dim N$.

Suppose that for each i, the set S_i is a sub manifold of M, then we can consider the restriction $f|S_{i1}$ of f to S_{i1} and define the singular set $S_{i1,i2}$ as the subset $S_{i2}(f|S_{i1})$ of S_{i1} . Again, if every set $S_{i1,i2}$ is a sub manifold of M, then the definition may be iterated. Thus, the set $S_{i1,\ldots,ik}$ is defined by induction as $S_{ik}(f|S_{i1,\ldots,ik-1})$. The index $\mathbb{j} = (i1, \ldots, ik)$ is called the symbol of the singularity. We will write $S_{\mathbb{j}}$ for $S_{i1,\ldots,ik}$.

For example the Whitney fold singular points of a mapping between surfaces and the Whitney umbrella of a mapping of a surface into a 3-manifold are Thom-Boardman singular points of the type $S_{1,0}$. From the Whitney description of singular points of a mapping between surfaces, it follows that the cusp singular points are of the type $S_{1,1,0}$.

Certainly, this natural definition makes sense only under heavy restrictions;
the singular set $S_{i1,\ldots,ik}$ can be defined only if the singularity stratum $S_{i1,\ldots,ik-1}$ is a sub manifold of the source manifold. By passing to jet spaces Boardman was able to extend the definition over all singular sets.

14.2 -Finite Jet Space

A singularity type of a mapping $f : M \to N$ at a point $x \in M$ depends on the behavior of the mapping f only in a small neighborhood of x. So, we pass to germs. A germ at a point $x \in M$ is an equivalence relation on mappings under which two mappings f_i, i = 1, 2, defined on a neighborhood of $x \in M$ represent the same germ at x if there is a possibly smaller neighborhood of x where the mappings f_1, f_2 coincide.

A k-jet is, by definition, a class of ~k-equivalence of germs. Two germs f and g at x are ~k equivalent if at the point x the mappings f and g have the same partial derivatives of order $\leq$ k. As partial derivatives involved, our definition implicitly assumes coordinate systems in neighborhoods of x and f(x) = g(x). It is easy to verify, however, that if in some coordinate systems f and g have the same partial derivatives of order $\leq$ k, then the same is true for any other choice of the coordinate systems.

If a k-jet α_X is represented by a mapping f at x, then we also say that α_X is a k-jet of the mapping f at x.

The set of all k-jets $J^k(M,N)$ is called the k-jet space of mappings of M into N. Let α_X be a k-jet at a point $x \in M$ represented by some mapping f.

If a coordinate system in a neighborhood of x and a coordinate system in a neighborhood of f(x) are fixed, then the k-jet α_X is determined by the Taylor polynomial of f at x of order k. In its turn, the set of polynomials of order k is naturally isomorphic to the finite dimensional Euclidean space as each polynomial characterized by the set of its coefficients. So, the k-jet space has a natural structure of a smooth manifold.

Formally, let u and v be coordinate covers of the manifolds M and N respectively. For each open set $U \in u$ and an open set $V \in v$ we define a subset W_{UV} of the k-jet space as the set of the jets α_X, $x \in U$, represented by mappings sending x into V . Note that the subsets $W = \{W_{UV}\}$, where U and V range over elements of u and v, cover the space of k-jets. Also, being isomorphic to the set of polynomials of order k, each of W_{UV} is isomorphic to a Euclidean space.

These isomorphism induce topologies, one for each W_{UV}, that coincide on intersections

$$W_{UV} \cap W_{U'V'}, U' \in u , V' \in v$$

Thus there is a natural topology on $J^k(M,N)$. Moreover, the cover W together with homeomorphisms from W_{UV} into the Euclidean space, $U \in u , V \in v$ defines a smooth structure on $J^k(M,N)$.

In fact the cover $\mathcal{w}$ not only helps to introduce a smooth structure on $J^k(M,N)$ but also allows us to introduce on $J^k(M,N)$ a structure of a smooth locally trivial bundle over $M \times N$. Indeed, $J^k(M,N)$ is covered by the sets $W_{UV} \in \mathcal{w}$ each of which is a trivial bundle over $U \times V$. We note that the bundle projection

$$J^k(M,N) \to M \times N$$

sends a k-jet α_x represented by a mapping f into the point $x \times f(x)$.

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14.3- Infinite Jet Space

A k-jet at a point x determines an l-jet for each l< k. Thus for each pair k, l with l< k we have a natural projection

$$\pi_l^k: J^k(M,N) \to J^l(M,N)$$

The jet space $J^\infty(M,N)$is a topological space defined as the inverse limit of the system $\{J^k(M,N), \pi_l^k\}$, though the jet space is infinite dimensional and we cannot
define a smooth structure on $J^\infty(M,N)$, still, using projections

$$\pi_k^\infty: J^\infty(M,N) \to J^k(M,N)$$

we may define on $J^\infty(M,N)$ smooth functions, tangent vectors and submanifolds.
We say that a function on the jet space is smooth if locally it is the composition of the projection onto some k-jet space and a smooth function on the k-jet space. Having defined smooth functions we may define a tangent vector at a jet α.
The set of germs $\mathcal{F}(\alpha)$of functions on $J^\infty(M,N)$ at the jet α is an algebra over R.
A differential operator $D_\alpha$at α is a correspondence

$$D_\alpha: \mathcal{F}(\alpha) \to \mathcal{F}(\alpha)$$

that is linear, i.e.,

$$D_\alpha(\mathrm{af+bg}) = \mathrm{a}D_\alpha(\mathrm{f}) + \mathrm{b}D_\alpha(\mathrm{g}),\ \mathrm{f, g} \in \mathcal{F}(\alpha),\ \mathrm{a, b} \in \mathrm{R},$$

and satisfy the Leibniz rule

$$D_\alpha\ (\mathrm{fg}) = D_\alpha\ (\mathrm{f}) + D_\alpha\ (\mathrm{g}), \qquad \mathrm{f, g} \in \mathcal{F}(\alpha)$$

We define a tangent vector at the jet α as a differential operator D_α. We may view a vector D_α as an infinite sequence of vectors $D_{\alpha k}$, k ∈ N, respectively tangent to the jet spaces J^k(M,N) at $\pi_k^\infty(\alpha)$ such that

$$\left(\pi_l^k\right)_*: D_{\alpha k} \to D_{\alpha l}\ \ k > l \ldots. (2.1)$$

Indeed, let α_k denote the k-jet $\pi_k^\infty(\alpha)$
By definition of $\mathcal{F}(\alpha)$,

$$\mathcal{F}(\alpha) = U_{k\in N}\ \mathcal{F}(\alpha_k)$$

where each $\mathcal{F}(\alpha_k)$ is identified with a subset of $\mathcal{F}(\alpha_{k+1})$ under the mapping induced by the projection
$\mathcal{F}(\alpha_{k+1}) - \mathcal{F}(\alpha_k)$. Given a vector D_α,its restrictions to $\mathcal{F}(\alpha_k)$ produce a sequence of operators $D_{\alpha k}$.

14.4 - Thom-Boardman Singularities

Given a smooth mapping : M → N, at each point of the manifold M we have an infinite jet of f. The correspondence that takes a point x into the jet of f at x is a mapping

$$\mathrm{j}\, f : \mathrm{M} \to J^\infty(M,N),$$

called the jet extension of f. Similarly for each k, we define the k-jet extension $\mathrm{j}^k f$ of f. If v is a vector at a point x of M, then the sequence of vectors d($\mathrm{j}^k f$)(v) satisfies the condition 2.1 and therefore defines a vector d(j f)(v) at j f(x).
Note that the map

$$\mathrm{d(j}\, f) : \mathrm{TM} \to \mathrm{T}J^\infty(\mathrm{M,N})$$

is injective. The union of images of d(jf) over all mappings M → N is called the total tangent bundle of the jet space and is denoted by D. Given a jet j f(x), we will use the injective homomorphism d(j f)|T_x to identify the plane T_xM with D|j f(x).

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Every 1-jet at a point x ∈ M determines a homomorphism

$$T_xM \rightarrow T_{f(x)}N,$$

where f is a germ at x representing the jet. Let y be a point of the jet bundle and $K_y \subset D_y$ the kernel of the homomorphism defined by the 1-jet component of y.
Boardman proved that for every i_1 the set

$$\sum\nolimits_{i_1} = \{y \in j^{\infty}(M,N) \mid \dim K_y = i_1\}$$

is a sub manifold of $j^{\infty}(M,N)$. Let j^r denote the set of r integers (i_1, ..., i_r) such that $i_1 \geq \cdots \geq i_r$. Suppose that the sub manifold $\sum_{j^{r-1}}$ has been already defined.

Then define

$$\sum\nolimits_{j^r} = \{y \in \sum\nolimits_{j^{r-1}} \mid \dim(k_y \cap T\sum\nolimits_{j^{r-1}}) = i_r\}$$

Boardman proved that for every symbol j^r the set $\sum_{j^r} is$ a submanifold of$j^{\infty}(M,N)$.
A mapping f is called a general position mapping if the section j f is transversal to every sub manifold $\sum_j$. By the Thom Strong Transversality Theorem every mapping can be approximated by a general position mapping.

Given a mapping : M → N, a point x ∈ M is a singularity of type I if the image j f(x) is in$\sum_j$.. As has been mentioned, for general position mappings,the definition of singularity types given by Boardman coincides with the naïve definition given in section 3.14.1.

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