ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 6 | Issue: 12 |December 2020 || Journal DOI: 10.36713/epra2013 || SJIF Impact Factor: 7.032 ||ISI Value: 1.188

A STUDY ON THE HYPERBOLA $Y^2=14x^2+1$

J.Shanthi[1]
[1]Assistant professor,
Department of mathematics,
SIGC,
Trichy

P.Deepalakshmi[2]
[2] PG Scholar,
Department of mathematics,
SIGC,
Trichy

M.A.Gopalan[3]
[3]Professor,
Department of mathematics,
SIGC,
Trichy

ABSTRACT

The binary quadratic equation $y^2=14x^2+1$ is considered and a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a few remarkable observations are illustrated.

KEYWORDS: ***Binary quadratic, hyperbola, integral solutions, pell equation.***

INTRODUCTION

Any non-homogeneous binary quadratic equation of the form y^2-Dx^2=1,where D is a given positive non-square integer, requiring integer solutions for x and y is called Pellian equation (also known pell-Fermat equation). In cartesian co-ordinates, the equation has the form of a hyperbola. The pellian equation has infinitely many distinct integer solutions as long as D is not a perfect square and the solutions are easily generated recursively from a single fundamental solution, namely, the solution with x, y positive integers of smallest possible size. One may refer [1-9] for a few choices of Pellian equations along with their corresponding integer solutions.

The solutions to Pellian equations have long been of interest to mathematicians. Even small values of D can lead to fundamental solutions which are quite large. For example, when D=61, the fundamental solution is (1766319049, 226153980). The above results motivated us to search for integer solutions to other choices of Pellian equation. This paper concerns with the Pellian equation $y^2=14x^2+1$, a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a few remarkable observations are illustrated.

METHOD OF ANALYSIS

The hyperbola represented by the non-homogeneous quadratic equation under consideration is

$$y^2=14x^2+1 \qquad (1)$$

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 6 | Issue: 12 |December 2020 || Journal DOI: 10.36713/epra2013 || SJIF Impact Factor: 7.032 ||ISI Value: 1.188

The smallest positive integer solution is $x_0=4$, $y_0=15$

If (x_n , y_n) represents the general solution of (**1**),then

$X_n=(1/2\sqrt{D})g_n$ (2)

$Y_n=(1/2)f_n$ (3)

where

$f_n=(15+4\sqrt{14})^{n+1}+(15-4\sqrt{14})^{n+1}$

$g_n=(15+4\sqrt{14})^{n+1}-(15-4\sqrt{14})^{n+1}$

A few numerical solutions to (**1**) are presented in table below:

Table: Numerical solutions

n	X_n	Y_n
0	4	15
1	120	449
2	3596	13455
3	107760	403151
4	3229204	12081075
5	9768360	362029099

Observations

- The values of x_n are even whereas the values of y_n are odd
- $x_n\equiv 0 \pmod 4$, $y_{2n}\equiv 0 \pmod 3$
- A few interesting relations among the solutions are given below:

  - $x_{n+2}-30x_{n+1}+x_n=0$
  - $y_{n+2} -30y_{n+1}+y_n=0$
  - $4y_n=x_{n+1}-15x_n$
  - $4y_{n+1} =15x_{n+1} -x_n$
  - $4y_{n+2} =449x_{n+1} -15x_n$
  - $y_{n+2}=112x_{n+1}+y_n$
  - $2 (y_{n+2} -y_{n+1})=217 x_{n+1}-7x_n$
  - $8y_{n+1}=x_{n+2}-x_n$
  - $56x_n=y_{n+1}-15y_n$
  - $56x_{n+1}=15y_{n+1}-y_n$
  - $56x_{n+2}= 449y_{n+1}-15y_n$
  - $y_{n+2} -56x_{n+1}=15y_{n+1}$

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 6 | Issue: 12 |December 2020 || Journal DOI: 10.36713/epra2013 || SJIF Impact Factor: 7.032 ||ISI Value: 1.188

- Expressions representing square integers:
  - $\frac{1}{2}(x_{2n+2} - 15x_{2n+1} + 4)$
  - $2(y_{n+1}+1)$

- Expressions representing cubical integers:
  - $\frac{1}{2}[x_{3n+3} - 15x_{3n+1} + 3x_{n+1} - 45x_n]$
  - $2(y3_{n+2} + 3y_n)$

- Expressions representing biquadratic integers:
  - $\frac{1}{2}(x_{4n+4} - 15x_{4n+3}) + 4f_n^2 - 2$
  - $\frac{1}{2}(x_{4n+4} - 15x_{4n+3}) + 4[\frac{1}{2}(x_{n+1} - 15x_n)]^2 - 2$
  - $2y_{4n+3} + 4f_n^2 - 2$
  - $2(y_{4n+3} + 4y_{n+1} + 3)$

- Employing linear combinations among the solutions, one obtains solutions to other choices of hyperbolas

Example1: Let $\mathbf{X} = x_{n+1} - 15x_n$, $\mathbf{Y} = x_n$

$$X^2 = 224Y^2 + 16$$

Note that (X, Y) satisfies the hyperbola

Example2: Let $\mathbf{X} = y_n$, $\mathbf{Y} = y_{n+1} - 15y_n$

$$224(X^2 - 1) = Y^2$$

Note that (X, Y) satisfies the hyperbola

- Employing linear combinations among the solutions, one obtains solutions to other choices of parabolas

Example3: Let $\mathbf{X} = x_{2n+2} - 15x_{2n+1} + 4$, $\mathbf{Y} = x_n$

$$X = 112Y^2 + 8$$

Note that (X, Y) satisfies the parabola

Example4: Let $\mathbf{X} = y_{n+1} + 1$, $\mathbf{Y} = x_n$

$$Y^2 = 112(X - 2)$$

Note that (X, Y) satisfies the parabola

- considering suitable values of X_n & Y_n, one generates 2^{nd} order Ramanujan numbers with base integers as real integers

For illustration, consider

$$X_1 = 120 = 1*120 = 2*60 = 3*40 = 12*10 = 6*20 \qquad (*)$$

Now, $1*120 = 2*60$

$\rightarrow (120+1)^2 + (60-2)^2 = (12-1)^2 + (60+2)^2$

$\rightarrow 121^2 + 58^2 = 119^2 + 62^2 = 18005$

1*120 = 3*40
$\rightarrow (120+1)^2 + (40-3)^2 = (120-1)^2 + (3+40)^2 = 16010$

1*120 = 12*10
$\rightarrow (120+1)^2 + (12-10)^2 = (20+1)^2 + (12+10)^2 = 14645$

1*120 = 6*20
$\rightarrow (1+120)^2 + (20-6)^2 = (20+6)^2 + (120-1)^2 = 14837$

2*60 = 3*40
$\rightarrow (2+60)^2 + (40-3)^2 = (60-2)^2 + (40+3)^2 = 5213$

3*40 = 12*10
$\rightarrow (3+40)^2 + (12-10)^2 = (12+10)^2 + (40-3)^2 = 1853$

3*40 = 6*20
$\rightarrow (3+40)^2 + (20-6)^2 = (20+6)^2 + (40-3)^2 = 2045$

Note:
2 * 60 = 12 *10 $\rightarrow 31^2-29^2=11^2 -1^2$
$\rightarrow 31^2+1^2=11^2+29^2 = 962$

2 * 60 = 6 * 20 $\rightarrow 31^2-29^2=13^2-7^2$
$\rightarrow 31^2+7^2=29^2+13^2 =1010$

12 *10 = 6 *20 $\rightarrow 11^2-1^2=13^2-7^2$
$\rightarrow 11^2+7^2=13^2+1^2=170$

Thus 18005,16010,14645,14837,5213,1853,2045,962,1010,170 represent 2^{nd} order Ramanujan numbers

- Considering suitable values of x_n & y_n, one generates 2^{nd} order Ramanujan numbers with base integers as guassian integers

For illustration, consider again x_1 represented by (*),

Now, 1*120 = 2*60 $\rightarrow (1+i120)^2 + (60-i2)^2 = -10803$
Also, 1*120 = 2*60 $\rightarrow (120+i)^2 + (2-i60)^2 =10803$

Note that -10803 & 10803 represent 2^{nd} order Ramanujan numbers with base integers as gaussian integers.

In a similar manner, other 2^{nd} order Ramanujan numbers are obtained.

REFERENCES

1. *Telang S.J., "Number Theory",Tata Mcgraw hill Publishing company ltd, New Delhi, (2000).*
2. *Thomas Koshy, "Elementary Number Theory Applications", Academic Press, (2005).*
3. *Gopalan M.A and Janaki G, "Observations on $y^2=3x^2+1$", Acta ciancia Indica, XXXIVM, No.2, (2008), 693-696.*
4. *Gopalan M.A and Yamuna R.S,"Remarkable Observations On The Binary Quadratic Equation", Impact J.Sci.Tech, Vol.4,No.4,(2010),61-65.*
5. *Gopalan M.A and Sangeetha G, "A Remarkable Observation On $y^2=10x^2+1$, Impact J.Sci.Tech.,Vol.4 No.4,(2010),103-106.*
6. *Gopalan M.A and Palanikumar R,"Observations On $y^2=12x^2+1$", Antarctia J.Math., 8(2)(2011),149-152.*
7. *Gopalan M.A and Geetha K, "Observations On The Hyperbola $y^2=18x^2+1$", RETELL, An Inter-disciplinary Research Journal Vol.13,No.1,(2012),81-83.*
8. *Gopalan M.A, Vijayalakshmi S, Usha Rani T.R and Malliga S, "Observations On $y^2=12x^2-3$", Bessel Journal Of Math, 2(3),(2012),153-158.*
9. *Gopalan M.A, Sumathi G and Vidhyalakshmi S, "Observations on $y^2=26x^2+1$", Bessel Journal Of Math, Vol .4, Issue 1, (2014),21-25.*