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HALF LOGISTIC MODIFIED EXPONENTIAL DISTRIBUTION: PROPERTIES AND APPLICATIONS

Arun Kumar Chaudhary[#1]
[1]Associate Professor,
Department of Management Science (Statistics)
Nepal Commerce Campus,
Tribhuwan University,
Nepal

Vijay Kumar[*2]
[2]Professor,
Department of Mathematics and Statistics,
DDU Gorakhpur University,
Gorakhpur, India

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ABSTRACT

In this study, we have introduced a three-parameter probabilistic model established from type I half logistic-Generating family called half logistic modified exponential distribution. The mathematical and statistical properties of this distribution are also explored. The behavior of probability density, hazard rate, and quantile functions are investigated. The model parameters are estimated using the three well known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises estimation (CVME) methods. Further, we have taken a real data set and verified that the presented model is quite useful and more flexible for dealing with a real data set.

KEYWORDS— ***Half-logistic distribution, Estimation, CVME ,LSE, , MLE***

1. INTRODUCTION

Since probability distributions are frequently used for the forecast of lifetime of components in various fields of applied sciences. Several lifetime distributions are introduced to model such types of data but existing distributions do not always produce an adequate fit. Hence in the last few decades generalizing distributions and exploring their flexibility and shapes are of interest to researchers. These new family of continuous distribution provides a better fit as compared to usual classical distributions and are acquired with the introduction of one or more additional shape parameter(s) to the baseline distribution.

Some of the well-known generating families are; Gamma-G (Zografos and Balakrishnan (2009)), Beta-generated (Beta-G) (Eugene et al. (2002)), exponentiated generalized class (Cordeiro et al. (2013)), gamma-exponentiated (Ristic and Balakrishnan (2011)), Exponential Half-Logistic family (Cordeiro et al. (2014), Kumaraswamy Weibull-G (Hassan and Elgarhy (2016)), (Cordeiro et al. (2015)) has introduced type first half-logistic family, Weibull-G (Bourguignon et al. (2014)), Garhy-G (Elgarhy et al. (2016)), additive Weibull- G (Hassan and Hemeda (2016)),exponentiated Weibull-G (Hassan and Elgarhy (2016)), Type II half logistic-G(TIIHLG) (Hassan et al. (2017)), exponentiated extended-G (Elgarhy et al. (2017), odd Frechet-G (Haq and Elgarhy (2018)), generalized additive Weibull-G (Hassan et al. (2017)), Muth-G (Almarashi and Elgarhy (2018)) Type II Half Logistic Rayleigh Distribution (Muhammad et al (2018)) and power Lindley-G (Hassan and Nassr (2018).

Cumulative distribution function (CDF) of the half logistic distribution given by (Balakrishnan, 1985) is,

$$F\left(y;\lambda\right)=\frac{1-e^{-\lambda y}}{1+e^{-\lambda y}}\quad ;y>0,\lambda>0$$

and its corresponding Probability density function(PDF) is

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$$f\left(y;\lambda\right)=\frac{2\lambda e^{-\lambda y}}{\left(1+e^{-\lambda y}\right)^{2}} \quad ;y>0,\lambda>0$$

The CDF and PDF of type I half logistic-G family which was introduced by Cordeiro et al. (2015) are,

$$F\left(x\right)=\frac{1-\left(1-G(x)\right)^{\lambda}}{1+\left(1-G(x)\right)^{\lambda}},\quad x,\lambda>0 \tag{1.1}$$

and

$$f\left(x\right)=\frac{2\lambda g(x)\left(1-G(x)\right)^{\lambda-1}}{\left[1+\left(1-G(x)\right)^{\lambda}\right]^{2}},\quad x,\lambda>0. \tag{1.2}$$

where G(x) and g(x) are CDF and PDF of baseline distribution.

The key objective of this article is to establish a new distribution using half-logistic generator, compare and investigate its various natures and behaviors. The remaining part of the article is structured as, In Section 2 we have explained the behavior of PDF and DCF, survival and hazard rate function, quantile function of the observed model. In Section 3 we have discussed the method of estimation namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises (CVM) methods. In section 4 we have estimated the model parameters using the above-mentioned methods and also for MLEs we have constructed asymptotic confidence interval of the estimate. The comparison of the flexibility of the observed model with some other models and application of the model using a real data set are illustrated in this section. In section 5 overall conclusions of the study are presented.

2. THE HALF LOGISTIC MODIFIED EXPONENTIAL DISTRIBUTION(HLME)

To develop HLME distribution we have used the modified exponential distribution as a baseline distribution which was introduced by (Rosaiah et al., 2007). The distribution function of the modified exponential distribution (MED) with two parameters $\left(\alpha,\lambda\right)>0$ is

$$F(x;\ \alpha,\ \lambda)=1-\exp\left\{-\alpha\ x\ \exp\left(\lambda x\right)\right\}\ ;(\alpha,\lambda)>0,\quad x>0 \tag{2.1}$$

where (α□□λ) are the shape parameters. The probability density function is

$$f(x;\ \alpha,\lambda)=\ \alpha\left(1+\lambda x\right)\exp\left\{\lambda x-\alpha\ x\ \exp\left(\lambda x\right)\right\}\quad ;(\alpha,\lambda)>0,\ x>0 \tag{2.2}$$

Now substituting (2.1) and (2.2) in (1.1) and (1.2) respectively we get the new half logistic modified exponential distribution whose CDF is obtained as

$$F\left(x\right)=\frac{1-\mathrm{e}^{-\alpha\lambda xe^{\beta x}}}{1+\mathrm{e}^{-\alpha\lambda xe^{\beta x}}};\quad \alpha,\beta,\lambda>0,\ x>0 \tag{2.3}$$

and corresponding PDF is

$$f\left(x\right)=\frac{2\alpha\lambda\left(1+\beta x\right)\mathrm{e}^{\beta x-\alpha\lambda xe^{\beta x}}}{\left(1+\mathrm{e}^{-\alpha\lambda xe^{\beta x}}\right)^{2}};\alpha,\beta,\lambda>0,\ x>0 \tag{2.4}$$

The reliability/survival function of HLME is

$$R(x)=\frac{2e^{-\alpha\lambda xe^{\beta x}}}{1+e^{-\alpha\lambda xe^{\beta x}}};\alpha,\beta,\lambda>0,\ x>0 \tag{2.5}$$

The hazard rate function (HRF) is

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$$h(x)=\frac{\alpha\lambda\left(1+\beta x\right)e^{\beta x}}{1+e^{-\alpha\lambda xe^{\beta x}}};\alpha,\beta,\lambda>0,\ x>0 \qquad (2.6)$$

The Quantile function

The quantile function related to the random variable's probability distribution, in probability statistics, defines the value of the random variable in a way that the probability of the variable being less than or equal to the value is equal to the probability assigned. It is also called the inverse cumulative distribution function or percent-point function. The definition of the pth quantile is following equation's real solution.

$$Q(p)=F^{-1}(p)$$

And we get quantile function by inverting (2.3) as

$$\ln x+\beta x-\ln\left[-\frac{1}{\alpha\lambda}\ln\left(\frac{1-p}{1+p}\right)\right]=0,\quad 0<p<1 \qquad (2.7)$$

For the generation of the random numbers of the HLME distribution, we suppose simulating values of random variable X with the CDF (2.3). Let U denote a uniform random variable in (0,1), then the simulated values of X is obtained by

$$\ln x+\beta x-\ln\left[-\frac{1}{\alpha\lambda}\ln\left(\frac{1-u}{1+u}\right)\right]=0\ ;\quad 0<u<1 \qquad (2.8)$$

Skewness and Kurtosis:

These measures are used mostly in data analysis to study the shape of the distribution or data set. The Bowley's skewness based on quartiles is,

$$S_k(B)=\frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},\ \text{and}$$

Coefficient of kurtosis based on octiles given by (Moors, 1988) is

$$K_u(Moor)=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}$$

Plots of probability density function and hazard rate function of HLME(α,β,λ) with different values of parameters are shown in Figure 1.

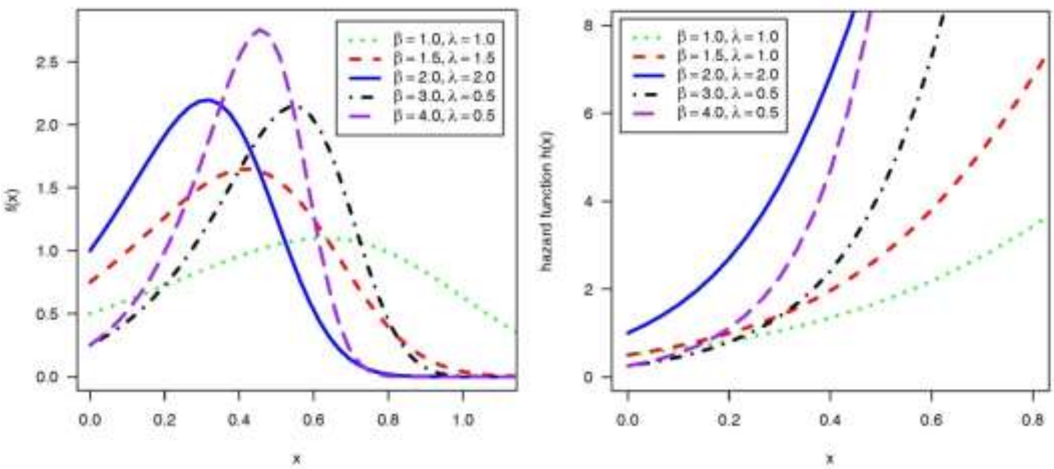

Figure 1. Plots of PDF (left panel) and hazard function (right panel) for α = 1 and different values of β and λ.

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3. METHOD OF PARAMETER ESTIMATION

3.1 Maximum Likelihood Estimation (MLE):

Here we discuss the maximum likelihood estimators (MLE's) of the HLME distribution. Let $\underline{x} = (x_1,\ldots,x_n)$ be a random sample of size 'n' from HLME(α, β, λ) then the likelihood function L(α, β, λ / $\underline{x}$) can be written as

$$L\left(\alpha,\beta,\lambda \mid \underline{x}\right)=2\alpha\lambda\prod_{i=1}^{n}\frac{\left(1+\beta x_i\right)\mathrm{e}^{\beta x_i-\alpha\lambda x_i e^{\beta x_i}}}{\left(1+\mathrm{e}^{-\alpha\lambda x_i e^{\beta x_i}}\right)^2} \tag{3.1.1}$$

Now the log-density of HLME is

$$l=\sum_{i=1}^{n}\beta x_i-\alpha\lambda\sum_{i=1}^{n}x_i e^{\beta x_i}+n\ln\left(2\alpha\lambda\right)+\sum_{i=1}^{n}\ln\left(1+\beta x_i\right)-2\sum_{i=1}^{n}\ln\left(1+\mathrm{e}^{-\alpha\lambda x_i e^{\beta x_i}}\right) \tag{3.1.2}$$

By differentiating (3.1.2) with respect to parameters and equating to zero we get maximum likelihood estimator of the parameters.

$$\begin{aligned}
\frac{\partial l}{\partial\alpha}&=\frac{1}{\alpha}-\lambda x e^{\beta x}+\frac{2\lambda x e^{\beta x}}{1+e^{\alpha\lambda x e^{\beta x}}}=0\\
\frac{\partial l}{\partial\beta}&=x-\alpha\lambda x^2 e^{\beta x}+\frac{x}{1+\beta x}+\frac{2\alpha\lambda x^2 e^{\beta x}}{1+e^{\alpha\lambda x e^{\beta x}}}=0\\
\frac{\partial l}{\partial\lambda}&=\frac{1}{\lambda}-\alpha x e^{\beta x}+\frac{x}{1+\beta x}+\frac{2\alpha x e^{\beta x}}{1+e^{\alpha\lambda x e^{\beta x}}}=0
\end{aligned} \tag{3.1.3}$$

Manually it is not possible to solve these nonlinear equations so we can use iterative techniques such as Newton-Raphson algorithm to calculate the estimated value of the parameters. The optim() function in R software can be used to solve them numerically. Let us denote the parameter vector by $\underline{\Omega}=(\alpha,\beta,\lambda)$ and the corresponding MLE of $\underline{\Omega}$ as $\hat{\underline{\Omega}}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$, then the asymptotic normality results in, $\left(\hat{\underline{\Omega}}-\underline{\Omega}\right)\to N_3\left[0,\left(I\left(\underline{\Omega}\right)\right)^{-1}\right]$ where $I\left(\underline{\Omega}\right)$ is the Fisher's information matrix given by,

$$I\left(\underline{\Omega}\right)=-\begin{pmatrix}
E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right)\\
E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right)\\
E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right)
\end{pmatrix}$$

In reality, since we do not know $\underline{\Omega}$, it is meaningless that the Maximum Likelihood Estimates has asymptotic variation $\left(I\left(\underline{\Omega}\right)\right)^{-1}$. Therefore by putting in the expected value of the parameters, we measure the asymptotic variance.The common technique is using the observed Fisher information matrix $O(\hat{\underline{\Omega}})$ as an estimate of the information matrix $I\left(\underline{\Omega}\right)$ is as follows,

$$O(\hat{\underline{\Omega}}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l}{\partial \alpha \partial \lambda} & \frac{\partial^2 l}{\partial \beta \partial \lambda} & \frac{\partial^2 l}{\partial \lambda^2} \end{pmatrix}_{\left|(\hat{\alpha},\hat{\beta},\hat{\lambda})\right.} = -H\left(\underline{\Omega}\right)_{\left|(\underline{\Omega}=\hat{\underline{\Omega}})\right.}$$

Where Hessian matrix is denoted by H

With the help of Newton-Raphson algorithm, maximizing the likelihood, gives observed information matrix. Therefore, the variance-covariance matrix is as follows,

$$\left[-H\left(\underline{\Omega}\right)_{\left|(\underline{\Omega}=\hat{\underline{\Omega}})\right.}\right]^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) \\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) & \mathrm{var}(\hat{\lambda}) \end{pmatrix}$$

Therefore from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α,β, λ can be produced as,

$$\hat{\alpha} \pm Z_{\alpha/2} SE(\hat{\alpha}) \text{ , } \hat{\beta} \pm Z_{\alpha/2} SE(\hat{\beta}) \text{ and } \hat{\lambda} \pm Z_{\alpha/2} SE(\hat{\lambda})$$

where upper percentile of standard normal variate is denoted by $Z_{\alpha/2}$

3.2. Method of Least-Square Estimation (LSE)

The weighted least square estimators and least-square estimators, proposed by Swain et al. (1988), is used for estimating the parameters of Beta distributions. In this article, the same technique is applied for the HLME distribution. The unknown parameters α, β, and λ's the least-square estimators of HLME distribution can be acquired by minimizing

$$S\left(X;\alpha,\beta,\lambda\right) = \sum_{j=1}^{n}\left[F(X_j) - \frac{j}{n+1}\right]^2 \tag{3.2.1}$$

with respect to unknown parameters α, β, and λ.

Let us consider $F(X_{(j)})$ represet the distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$, where $\{X_1, X_2, \ldots, X_n\}$ denote random sample of size n from a distribution function F(.). Thus, in this case, the least square estimators of α, β, and λ say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ respectively, can be acquired by minimizing

$$S\left(X;\hat{\alpha},\hat{\beta},\hat{\lambda}\right) = \sum_{j=1}^{n}\left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{j}{n+1}\right]^2 \tag{3.2.2}$$

with respect to α, β, and λ. Differentiating (3.2.2) with respect to α, β, and λ we get the following three nonlinear equations as

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$$\frac{\partial S}{\partial \alpha} = -4\lambda \sum_{j=1}^{n} \left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{j}{n+1} \right] \frac{x_j\, \mathrm{e}^{\alpha\lambda x_j e^{\beta x_j}+\beta x_j}}{\left(1+\mathrm{e}^{\alpha\lambda x_j e^{\beta x_j}}\right)^2}$$

$$\frac{\partial S}{\partial \beta} = -4\alpha\lambda \sum_{j=1}^{n} \left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{j}{n+1} \right] \frac{x_j^2\, \mathrm{e}^{\alpha\lambda x_j e^{\beta x_j}+\beta x_j}}{\left(1+\mathrm{e}^{\alpha\lambda x_j e^{\beta x_j}}\right)^2}$$

$$\frac{\partial S}{\partial \lambda} = -4\alpha \sum_{j=1}^{n} \left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{j}{n+1} \right] \frac{x_j^2\, \mathrm{e}^{\alpha\lambda x_j e^{\beta x_j}+\beta x_j}}{\left(1+\mathrm{e}^{\alpha\lambda x_j e^{\beta x_j}}\right)^2}$$

The unknown parameters' weighted least square estimators can be obtained by minimizing

$$S\left(X;\alpha,\beta,\lambda\right) = \sum_{j=1}^{n} w_j \left[F(X_{(j)}) - \frac{j}{n+1} \right]^2$$

with respect to α, β, and λ. The weights w_j are $w_j = \frac{1}{V(X_{(j)})} = \frac{\left(n+1\right)^2 \left(n+2\right)}{j\left(n-j+1\right)}$

Hence, the weighted least square estimators of α, β, and λ respectively, can be acquired by minimizing,

$$S\left(X;\alpha,\beta,\lambda\right) = \sum_{j=1}^{n} \frac{\left(n+1\right)^2 \left(n+2\right)}{j\left(n-j+1\right)} \left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{j}{n+1} \right]^2 \qquad (3.2.3)$$

with respect to α, β, and λ.

3.3. Method of Cramer-Von-Mises (CVM)

We are interested in Cramér-Von-Mises type minimum distance estimators, (Macdonald 1971) because it provides empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The CVM estimators of α, β, and λ are acquired by minimizing the function

$$C\left(\alpha,\beta,\lambda\right) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F\left(x_{i:n} \,|\, \alpha,\beta,\lambda\right) - \frac{2i-1}{2n} \right]^2 \qquad (3.3.1)$$

$$= \frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{2i-1}{2n} \right]^2$$

Differentiating (3.3.1) with respect to α, β, and λ we get the following three nonlinear equations as

$$\frac{\partial C}{\partial \alpha} = -4\lambda \sum_{j=1}^{n} \left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{2i-1}{2n} \right] \frac{x_j\, \mathrm{e}^{\alpha\lambda x_j e^{\beta x_j}+\beta x_j}}{\left(1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}\right)^2}$$

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$$\frac{\partial C}{\partial \beta} = -4\alpha\lambda \sum_{j=1}^{n} \left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{2i-1}{2n} \right] \frac{x_j^2\, \mathrm{e}^{\alpha\lambda x_j e^{\beta x_j} + \beta x_j}}{\left(1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}\right)^2}$$

$$\frac{\partial C}{\partial \lambda} = -4\alpha \sum_{j=1}^{n} \left[\frac{1-\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}}{1+\mathrm{e}^{-\alpha\lambda x_j e^{\beta x_j}}} - \frac{2i-1}{2n} \right] \frac{x_j^2\, \mathrm{e}^{\alpha\lambda x_j e^{\beta x_j} + \beta x_j}}{\left(1+\mathrm{e}^{\alpha\lambda x_j e^{\beta x_j}}\right)^2}$$

4. APPLICATION OF REAL DATASET

In this portion, we demonstrate the applicability of HLME distribution with the help of a real dataset used by earlier researchers. The following data shows the service times of 63 Aircraft Wind shield (Kundu & Raqab, 2009) and listed as follows

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140

From Figure 2 it is seen that the estimated values of the parameters via MLE method are unique.

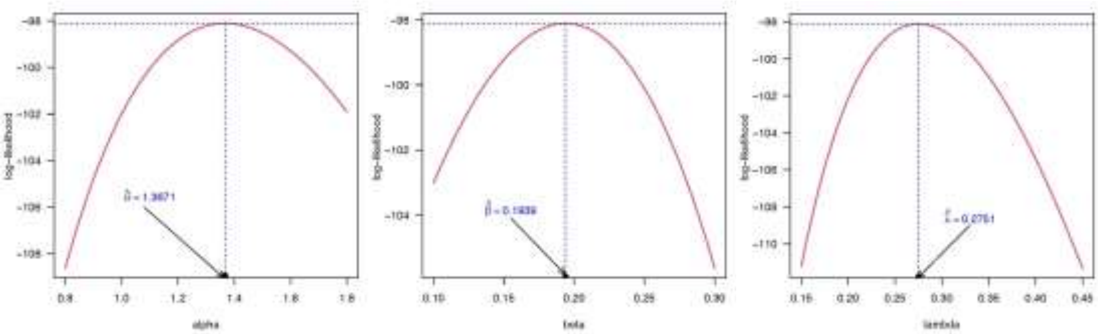

Figure 2. Profile log-likelihood functions of α, β and λ.

For checking model's validity is checked by computing the Kolmogorov-Smirnov (KS) distance between the fitted distribution function and the empirical distribution function when the parameters are acquired from maximum likelihood method. The KS plot is presented in Figure 3 demonstrates that the estimated exponential extension model provides excellent fitt to the given data. By maximizing the likelihood function in (3.1), we have computed the maximum likelihood estimates directly by using R software (R Core Team, 2020) and (Rizzo, 2008). We have obtained $\hat{\alpha}$ = 1.3671, $\hat{\beta}$=0.1939, $\hat{\lambda}$=0.2751 and corresponding Log-Likelihood value is l= -98.10808. In Table 1 we have demonstrated the MLE's with their standard errors (SE) for α, β, and λ.

Table 1

MLE and SE for α, β and λ

Parameter	MLE	SE	t-value	Pr(>t)
alpha	1.3671	14.9193	0.092	0.9269
beta	0.1939	0.0629	3.084	0.00204
lambda	0.2751	3.0028	0.092	0.92701

In Table 2 we have displayed the estimated value of the parameters, log-likelihood, AIC, BIC and AICC for the MLE, LSE and CVE methods. It is found that the MLEs are quite better among these three estimation methods.

Table 2

Estimated parameters, log-likelihood, AIC, BIC, AICC and HQIC

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-LL	AIC	BIC	AICC	HQIC
MLE	1.3671	0.1939	0.2751	98.1081	202.2162	208.6456	202.6229	204.7449
LSE	1.5304	0.2193	0.2308	98.2072	202.4145	208.8439	202.8212	204.9432
CVE	1.5290	0.2378	0.2220	98.4171	202.8342	209.2636	203.2410	205.3630

In Table 3 we have displayed the value of the test statistic of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) to assess the goodness of fit of MLE, LSE and CVE estimation methods.

Table 3

The goodness-of-fit statistics and their corresponding p-value

Method	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
MLE	0.0665(0.9254)	0.0351(0.9578)	0.2425(0.9741)
LSE	0.0609(0.9622)	0.0304(0.9756)	0.2561(0.9670)
CVE	0.0590(0.9714)	0.0294(0.9788)	0.2932(0.9430)

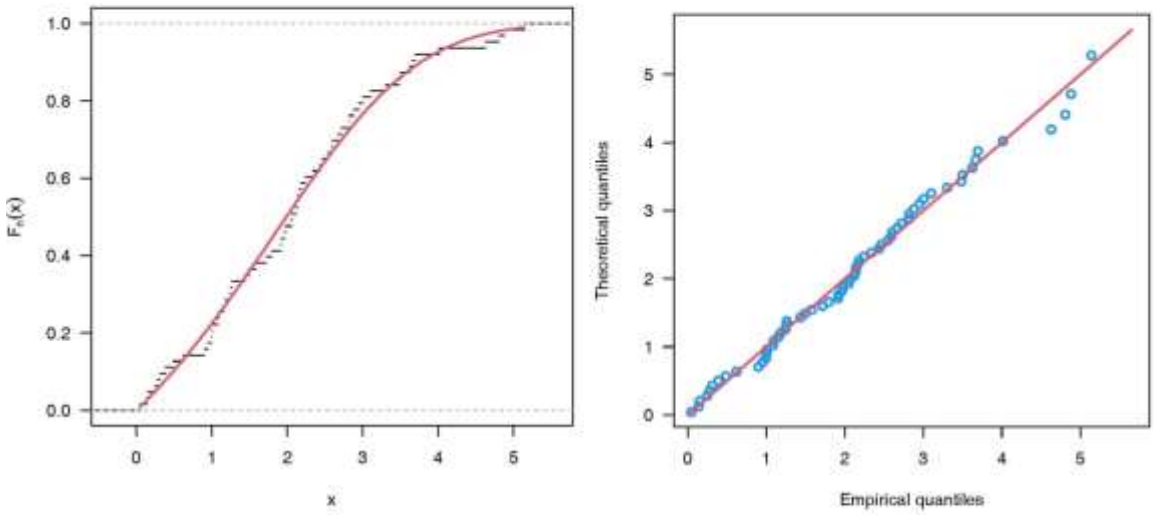

Figure 3. The graph of the CDF (left panel) and the Q-Q plot (right panel) of HLME distribution.

To illustrate the goodness of fit of the HLME distribution, we have taken some well known distributions for comparison purpose which are listed below,

I. Exponentiated Exponential Poisson (EEP) distribution

The probability density function of EEP (Ristić & Nadarajah, 2014) can be expressed as

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$$f(x)=\frac{\alpha\beta\lambda}{\left(1-e^{-\lambda}\right)}e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\exp\left\{-\lambda\left(1-e^{-\beta x}\right)^{\alpha}\right\} \quad ; x>0, \alpha>0, \lambda>0$$

II. Generalized Exponential Extension (GEE) distribution

The PDF of GEE distribution has introduced by (Lemonte, 2013) with parameters α , β and λ is

$$f_{GEE}\left(x;\alpha,\beta,\lambda\right)=\alpha\beta\lambda\left(1+\lambda x\right)^{\alpha-1} exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}$$
$$\left[1-exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}\right]^{\beta-1} \quad ; \; x\geq 0.$$

III. Generalized Gompertz (GGZ) distribution

The PDF of GGZ distribution has introduced by (El-Gohary et al., 2013) is

$$f_{GG}\left(x\right)=\theta\lambda e^{\alpha x}e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\right]^{\theta-1} \; ; \; \lambda\theta>0, \alpha\geq 0, x\geq 0$$

IV. Exponential Extension (EE) distribution

The density of exponential extension (EE) distribution (Nadarajah & Haghighi, 2011) with parameters α and λ is

$$f_{EE}(x)=\alpha\,\lambda\left(1+\lambda x\right)^{\alpha-1}\exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\} \quad ;x\geq 0, \;\alpha>0, \;\lambda>0.$$

In order to check the goodness of fit of a given distribution we generally use the PDF and CDF plot. To get the additional information we have to plot CDF and P-P plots. Particularly the CDF plot may give information about the lack-of-fit at the tails of the distribution, whereas the P-P plot emphasizes the lack-of-fit. From Figure 4 it is proven that the HLME model fits the data very well.

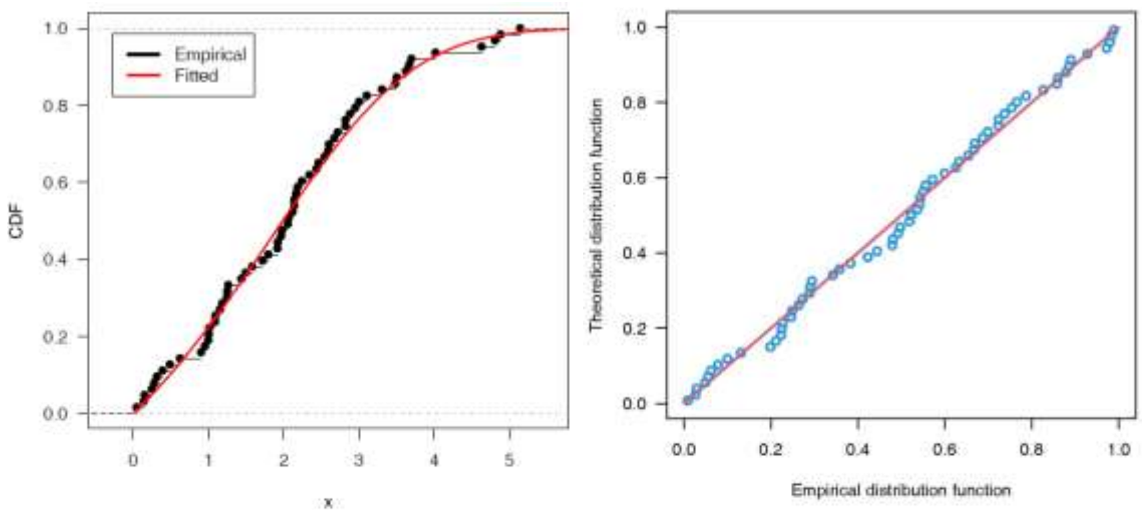

Figure 4. The graph of empirical distribution function with estimated distribution function (left panel) and P-P plot (right panel) of HLME distribution.

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For the assessment of potentiality of the proposed model we have calculated the Bayesian information criterion (BIC), Akaike information criterion (AIC),Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4. It is observed that the HLME distribution have the smallest value of log likelihood, AIC, BIC, CAIC and HQIC.

Table 4

Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
HLME	98.1081	202.2162	208.6456	202.6229	204.7449
EEP	103.5468	213.0936	219.5230	213.5004	215.6224
GEE	98.6627	203.3254	209.7548	203.7322	205.8541
GGZ	98.2316	202.4633	208.8927	202.8701	204.9920
EE	100.1167	204.2335	208.5197	204.4270	205.9193

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of HLME and some selected distributions are presented in Figure 5.

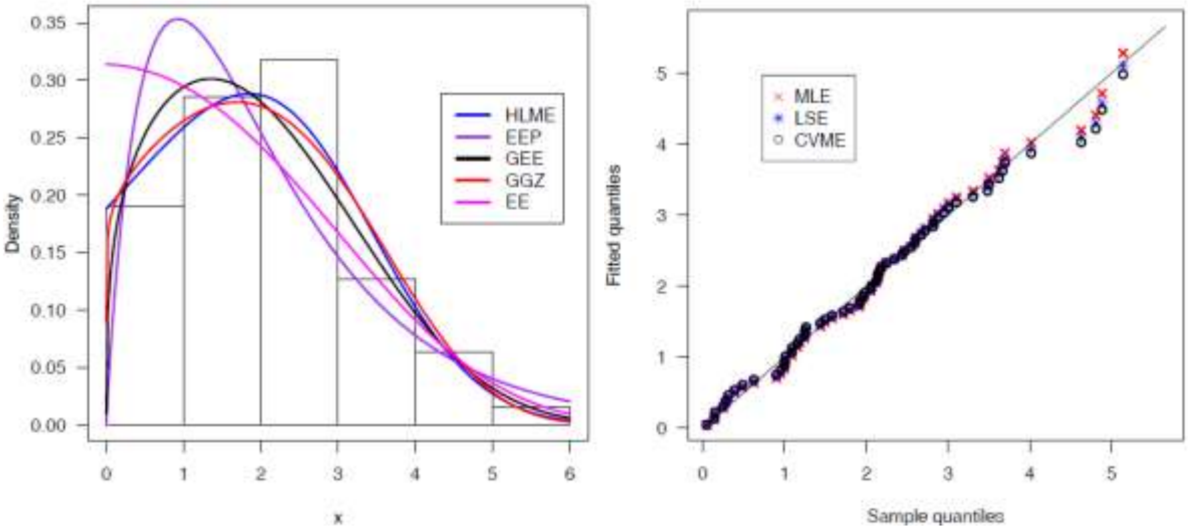

Figure 5. The Histogram and the density function of fitted distributions (left panel) of HLME distribution and fitted quantiles of estimation methods MLE, LSE and CVME (right panel).

For comparing the goodness-of-fit of the HLME distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 5. It is observed that the HLME distribution has the minimum value of the test statistic and higher p-value thus we derive the conclusion that the HLME distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 5

The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
HLME	0.0665(0.9254)	0.0351(0.9578)	0.2425(0.9741)
EEP	0.1431(0.1373)	0.2298(0.2167)	1.3052(0.2306)
GEE	0.0958(0.5771)	0.0684(0.7633)	0.4040(0.8442)

GGZ	0.0694(0.9009)	0.0428(0.9197)	0.2890(0.9460)
EE	0.1446(0.1296)	0.2949(0.1397)	1.3923(0.2044)

5. CONCLUSIONS

In this study, we have studied the three-parameter half logistic modified exponential (HLME) distribution. We have demonstrated the PDF, the CDF, and the shapes of the hazard function. The PDF's shape of the HLME distribution is unimodal and positively skewed, while the hazard function of the HLME distribution is increasing. The P-P and Q-Q plots demonstrates that the purposed distribution is quite better for fitting the real dataset. Finally, using a real data set we explore the maximum likelihood estimates of the parameters and their corresponding confidence interval. Further, we also investigate the least square estimates, and Cramer-Von-Mises methods of estimation for estimating the parameters. We have compared the potentiality of the proposed distribution with some other distributions by employing goodness of fit test and found that the HLME distribution fits quite better than others taken in consideration.

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