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ON THE HOMOGENEOUS QUADRATIC DIOPHANTINE EQUATION WITH THREE UNKNOWNS

$$4x^2-12xy+21y^2=13z^2$$

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ABSTRACT

The ternary quadratic equation given by $4x^2-12xy+21y^2=13z^2$ ***is considered and searched for its many different integer solution . Five different choices of integer solution of the above equations are presented .A few interesting relations between the solutions and special polygonal numbers are presented.***

KEY WORDS: ***ternary quadratic, integer solutions***
MSC subject classification :11D09

1. INTRODUCTION

The Diophantine equation offer an unlimited field for research due to their variety [1-3].In particular ,one may refer [4-15] for quadratic equations with three unknowns . This communication concerns with yet another interesting equation $4x^2-12xy+21y^2=13z^2$ representing homogeneous equation with three for determining its infinitely Many non -zero integral points. Also , few interesting relations among the solutions are presented.

2. NOTATIONS

- $t_{m,n}=n^{th}$ term of a regular polygon with m sides.

  $$=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

- $\Pr_n=$ pronic number of rank n

  $$=n(n+1)$$

3. METHOD OF ANALYSIS

The Quadratic Diophantine equation with three unknowns to be solved is given by,

$$4x^2 - 12xy + 21y^2 = 13z^2 \qquad (1)$$

substituting

$$2x - 3y = U \qquad (2)$$

in (1) we get,

$$U^2 + 12y^2 = 13z^2 \qquad (3)$$

(3) is solved through different approaches and the different patterns of solutions (1) obtained are presented below.

PATTERN:1

Assume

$$Z = \left(a^2 + 12b^2\right)$$

(3) can also be written as,

$$U^2 + 12Y^2 = 13Z^2$$

write '13' as,

$$13 = (1 + i\sqrt{12})(1 - i\sqrt{12})$$

in equation (3),we get,

$$\left(U + i\sqrt{12}y\right)\left(U - i\sqrt{12}y\right) = \left(1 + i\sqrt{12}y\right)\left(1 - i\sqrt{12}y\right)\left(a + i\sqrt{12}b\right)^2\left(a + i\sqrt{12}b\right)^2$$

Equating positive terms

$$\left(U+i\sqrt{12}y\right)=\left(1+i\sqrt{12}\right)\left(a^2-12b^2+i\sqrt{12}ab\right)$$
$$=\left(a^2-b^2-24ab\right)+i\sqrt{12}\left(a^2-12b^2+2ab\right)$$

Equating real and imaginary parts

$$\left.\begin{array}{l} U=a^2-b^2-24ab \\ y=a^2-b^2+2ab \end{array}\right\} \quad (4)$$

From (2)

$$2x-3y=U$$

we obtain the non-zerodistinct integral solution of (1) as

$$x(a,b)=2a^2-24b^2-9ab$$

$$y(a,b)=a^2-12b^2+2ab$$

$$z(a,b)=a^2+12b^2$$

PROPERTIES:

[1] $y(a,1)+z(a,1)-t_{6,a}=3a$

[2] $y(a,a+1)+z(a,a+1)-2pr_a+2t_{4,a}=(2a)^2$ is perfect square

[3] $x(1,b)+y(1,b)+29t_{4,b}+7pr_b=3$

PATTERN:2

‘13’ can also be written as

$$13 = \frac{\left(14 + i\sqrt{12}\right)\left(14 - i\sqrt{12}\right)}{2} \tag{5}$$

Substituting (5) in (3) and employing the method of factorization,

we get,

$$\left(U + i\sqrt{12}y\right)\left(U - i\sqrt{12}y\right) = \frac{\left(14 + i\sqrt{12}y\right)\left(14 - i\sqrt{12}y\right)}{4^2}\left(a + i\sqrt{12}b\right)^2\left(a + i\sqrt{12}b\right)^2$$

Consider the positive factor,

$$\left(U + i\sqrt{12}y\right) = \frac{\left(14 + i\sqrt{12}\right)}{4}\left(a + i\sqrt{12}b\right)^2$$

$$= \left(\frac{14 + i\sqrt{12}}{4}\right)\left(a^2 - 12b^2 + i2ab\sqrt{12}\right)$$

$$= \frac{1}{4}\left[14\left(a^2 - 12b^2 - 24ab\right) + i\sqrt{12}\left(a^2 - 12b^2 + 28ab\right)\right]$$

Equating real and imaginary parts of the above equation, we get

$$U = \frac{14a^2 - 168b^2 - 24ab}{4}$$

$$y = \frac{a^2 - 12b^2 + 28ab}{4}$$

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From (2)

$$2x-3y=U$$

$$x=\frac{17a^2-204b^2+60ab}{2}$$
$$y=\frac{a^2-12b^2+28ab}{4}$$
$$z=a^2+12b^2$$

Assume a =4A, b =4B in the above equations, we obtain the non-zero distinct integral solution of (2.1) as

$$x(A,B)=34A^2-408B^2+120AB$$
$$y(A,B)=4A^2-48B^2+112AB$$
$$Z(A,B)=16(A^2+12B^2)$$

PROPERTIES:

- $x(A,1)+y(A,1)+230t_{4,A}-232pr_A+448=36A^2$ is a perfect square.
- $3[y(1,B)+2(1,B)-112pr_B-20]$ is a nasty number.
- $x(A,1)+y(A,1)+z(A,1)+54pr_a\equiv 0(\mathrm{mod}\,2)$

PATTERN:3

1 can be written as,

$$1=\frac{(1+2i\sqrt{12})(1-i\sqrt{12})}{7^2} \quad (6)$$

substituting (6) in (3) we get,

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$$\left(U+i\sqrt{12}y\right)\left(U-i\sqrt{12}y\right)=\left(1+i\sqrt{12}\right)\left(1-i\sqrt{12}\right)\left(a+i\sqrt{12}b\right)^2\left(a-i\sqrt{12}b\right)^2\frac{\left(1+2i\sqrt{12}\right)\left(1-2i\sqrt{12}\right)}{7^2}$$

Equating positive terms we get,

$$\left(U+i\sqrt{12}y\right)=\left(1+i\sqrt{12}\right)\left(a+i\sqrt{12}b\right)^2\frac{\left(1+2i\sqrt{12}\right)}{7}$$

$$\frac{1}{7}\left[\left(-23a^2+276b^2-72ab\right)+i\sqrt{12}\left(3a^2-36b^2-46ab\right)\right]$$

Equating real and imaginary parts we get,

$$\left.\begin{aligned}U&=\frac{1}{7}\left(-23a^2+276b^2-72ab\right)\\ y&=\frac{1}{7}\left(3a^2-36b^2-46ab\right)\end{aligned}\right\}$$

From (2) (7)

$U=2x-3y$

$$2x=\frac{1}{7}\left(14a^2+168b^2-210ab\right)$$

$$\Rightarrow x=\frac{1}{7}\left(-7a^2+84b^2-105ab\right)$$

$$y=\frac{1}{7}\left(3a^2-36b^2-46ab\right)$$

$$z=a^2+12b^2$$

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Assume $a = 7A, b = 7B$ in the above equation , we obtain the non-zero distinct integral solution of (2.1) as,

$$x(A,B) = 49\left(-A^2 + 12B^2 - 15AB\right)$$

$$y(A,B) = 7\left(3A^2 - 36B^2 - 46AB\right)$$

$$Z(A,B) = 49\left(A^2 + 12B^2\right)$$

PROPERTIES:

[1] $y(A,1) + z(A,1) - 392t_{4,A} + 322\,\mathrm{Pr}_A = 336$

[2] $x(A,1) + y(A,1) - 1029t_{4,A} + 1057\,\mathrm{Pr}_A \equiv 0 \pmod 3$

[3] $x(A,1) + z(A,1) + 735\,\mathrm{Pr}_A - 735t_{4,A} \equiv 0 \pmod 5$

PATTERN 4

13 can also be written as,

$$13 = \frac{\left(14 + i\sqrt{12}\right)\left(14 - i\sqrt{12}\right)}{4^2}$$

Write 1 as,

$$1 = \frac{\left(1 + 2i\sqrt{12}\right)\left(1 - 2i\sqrt{12}\right)}{7^2}$$

Equating positive terms we get,

$$\left(U+i\sqrt{12}y\right)=\frac{\left(14+i\sqrt{12}\right)}{4}\left(\frac{1+2i\sqrt{12}}{7}\right)\left(a+i\sqrt{12}b\right)^2$$

$$=\frac{1}{28}\left[\left(-10a^2+120b^2-696ab\right)+i\sqrt{12}\left(29a^2-348b^2-20ab\right)\right]$$

Equating real and imaginary parts,

$$U=\frac{1}{28}\left(-10a^2+120b^2-696ab\right)$$

$$y=\frac{1}{28}\left(29a^2-348b^2-20ab\right) \qquad (8)$$

From (2)

$$U=2x-3y$$

$$x=\frac{1}{56}\left(77a^2-924b^2-756ab\right)$$

$$y=\frac{1}{28}\left(29a^2-348b^2-20ab\right)$$

$$z=a^2+12b^2$$

Assume $a=56A, b=56B$ in the above equation , we obtain the non-zero distinct integer solution of(1) are given by,

$$x(A,B)=56\left(77A^2-924B^2-756AB\right)$$

$$y(A,B)=112\left(29A^2-348B^2-20AB\right)$$

$$z(A,B)=3136\left(A^2+12B^2\right)$$

PROPERTIES

[1] $x(A,1)+y(A,1)-52136t_{4,A}+44576pr_A \equiv 0(\mathrm{mod}2)$

[2] $x(A,1)-y(A,1)-41160t_{4,A}+40096pr_A=-12768$

[3] $x(A,1)+z(A,1)-49784t_{4,A}+42336pr_A=-14112$

PATTERN 5

Equation (3) can be written as,

$$U^2+12y^2=12z^2+z^2$$

$$U^2-z^2=12(z^2-y^2)$$

$$(U+z)(U-z)=12(z+y)(z-y) \tag{9}$$

Case1

Equation (9) can also be written as,

$$\left(\frac{U+z}{U-z}\right)=12\frac{(z-y)}{(U-z)}=\frac{\alpha}{\beta}$$

Which is equivalent to the system of double equation as,

$$\left.\begin{aligned}\beta U+z(\beta-\alpha)-\alpha y=0\\ -\alpha U+z(12\beta+\alpha)-12\beta y=0\end{aligned}\right\} \tag{10}$$

Solving (10) by the method of cross multiplication, we get

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$$\left.\begin{aligned} U &= -\alpha^2 - 12\beta^2 + 24\alpha\beta \\ y &= -\alpha^2 + 12\beta^2 + 2\alpha\beta \\ z &= \alpha^2 + 12\beta^2 \end{aligned}\right\} \quad (11)$$

Substituting (11) in (2), the non-zero distinct integer solution of(1) are given by,

$$x(\alpha,\beta) = -\alpha^2 + 12\beta^2 + 15\alpha\beta$$

$$y(\alpha,\beta) = -\alpha^2 + 12\beta^2 + 2\alpha\beta$$

$$z(\alpha,\beta) = \alpha^2 + 12\beta^2$$

PROPERTIES

[1] $x(\alpha,1) + y(\alpha,1) + 19t_{4,\alpha} - 17\Pr_\alpha \equiv 0 \pmod 2$

[2] $x(\alpha,1) - y(\alpha,1) + 13t_{4,\alpha} - 13\Pr_\alpha = 0$

[3] $x(\alpha,1) + z(\alpha,1) + 15t_{4,\alpha} - 15\Pr_\alpha \equiv 0 \pmod 3$

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Case 2

Equation (9) can also be written as ,

$$\frac{(U+z)}{3(z+y)} = 4\frac{(z-y)}{(U-z)} = \frac{\alpha}{\beta}$$

Which is equivalent to the system of double equation as,

$$\left.\begin{aligned} -\alpha U + z(\alpha+4\beta) - 4\beta y = 0 \\ \beta U + z(\beta - 3\alpha) - 3\alpha y = 0 \end{aligned}\right\} \quad (12)$$

Solving (12) by method of cross multiplication, we get

$$\left.\begin{aligned} U &= 3\alpha^2 - 4\beta^2 + 24\alpha\beta \\ y &= -3\alpha^2 + 4\beta^2 + 2\alpha\beta \\ z &= 3\alpha^2 + 4\beta^2 \end{aligned}\right\} \quad (13)$$

Substituting (13) in (3) ,the non-zero distinct integer solution of (1) are given by,

$$x(\alpha,\beta) = -3\alpha^2 + 4\beta^2 + 15\alpha\beta$$

$$y(\alpha,\beta) = -3\alpha^2 + 4\beta^2 + 2\alpha\beta$$

$$z(\alpha,\beta) = 3\alpha^2 + 4\beta^2$$

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PROPERTIES

[1] $x(\alpha,1)+y(\alpha,1)+23t_{4,\alpha}-17\Pr_{\alpha}\equiv 0(\bmod 2)$

[2] $x(\alpha,1)-y(\alpha,1)+13t_{4,\alpha}-13\Pr_{\alpha}=0$

[3] $x(\alpha,1)+z(\alpha,1)+15t_{4,\alpha}-15\Pr_{\alpha}=8$

Case: 3

(9) can be written in the form of ratio as,

$$6\frac{(U+z)}{(z+y)}=2\frac{(z-y)}{(U-z)}=\frac{\alpha}{\beta}$$

which is equivalent to the system of double equation is,

$$\left.\begin{aligned}\beta U+z(\beta-6\alpha)-6\alpha y=0\\-\alpha U+z(2\beta+\alpha)-2\beta y=0\end{aligned}\right\} \qquad (14)$$

Solving (14) by method of cross multiplication, we get

$$U=6\alpha^2-2\beta^2+24\alpha\beta$$

$$y=-6\alpha^2+2\beta^2+2\alpha\beta \qquad (15)$$

$$z=6\alpha^2+2\beta^2$$

substituting (15) in (2), the non-zero distinct integer solution of (1) are given by,

$$x(\alpha,\beta) = -6\alpha^2 + 2\beta^2 + 15\alpha\beta$$

$$y(\alpha,\beta) = 6\alpha^2 + 2\beta^2 + 2\alpha\beta$$

$$z(\alpha,\beta) = 6\alpha^2 + 2\beta^2$$

PROPERTIES

[1] $x(\alpha,1) + y(\alpha,1) + 29t_{4,\alpha} - 17\Pr_\alpha \equiv 0 \pmod 2$

[2] $x(\alpha,1) - y(\alpha,1) + 13t_{4,\alpha} - 13\Pr_\alpha = 0$

[3] $x(\alpha,1) + z(\alpha,1) + 15t_{4,\alpha} - 15\Pr_\alpha = 4$

Case: 4

Equation (9) can also be written as

$$2\frac{(U+z)}{(z+y)} = 6\frac{(z-y)}{(U-z)}$$

Which is equivalent to the system of double equations as,

$$\beta U + z(\beta - 2\alpha) - 2\alpha y = 0$$

$$-\alpha U + z(6\beta + \alpha) - 6\beta y = 0$$

Solving above equation by method of cross multiplication, we get

$$U = 2\alpha^2 - 6\beta^2 + 24\alpha\beta \qquad (16)$$

$$y = -2\alpha^2 + 6\beta^2 - 2\alpha\beta$$

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$$z = 2\alpha^2 + 6\beta^2$$

Substituting (16) in (2), the non-zero distinct integer solution of (1) are given by,

$$x(\alpha,\beta) = -2\alpha^2 + 6\beta^2 + 15\alpha\beta$$

$$y(\alpha,\beta) = -2\alpha^2 + 6\beta^2 + 2\alpha\beta$$

$$z(\alpha,\beta) = 2\alpha^2 + 6\beta^2$$

PROPERTIES

[1] $x(\alpha,1) + y(\alpha,1) + 21t_{4,\alpha} - 17\Pr_\alpha \equiv 0 \pmod 3$

[2] $x(\alpha,1) - y(\alpha,1) + 13t_{4,\alpha} - 13\Pr_\alpha = 0$

[3] $x(\alpha,1) + z(\alpha,1) + 15t_{4,\alpha} - 15\Pr_\alpha = 12$

Case: 5

Equation (9) can be written as,

$$4\frac{(U+z)}{(z+y)} = 3\frac{(z-y)}{(u-z)} = \frac{\alpha}{\beta}$$

which is equivalent to the to double equations as,

$$\beta U + z(\beta - 4\alpha) - 4\alpha\beta = 0$$

$$-\alpha U + z(\alpha + 3\beta) - 3\beta y = 0 \qquad (17)$$

Solving (17) by method of cross multiplication, we get

$$U = 4\alpha^2 - 3\beta^2 + 24\alpha\beta$$

$$y = -4\alpha^2 + 3\beta^2 + 2\alpha\beta$$

$$z = 4\alpha^2 + 3\beta^2$$

(18)

Substituting (18) in (2), the non-zero distinct integer solution of (1) are given by,

$$x(\alpha,\beta) = -4\alpha^2 + 3\beta^2 + 15\alpha\beta$$

$$y(\alpha,\beta) = -4\alpha^2 + 3\beta^2 + 2\alpha\beta$$

$$z(\alpha,\beta) = 4\alpha^2 + 3\beta^2$$

PROPERTIES

[1] $x(\alpha,1) + y(\alpha,1) + 25t_{4,\alpha} - 17\mathrm{Pr}_\alpha \equiv 0 \pmod 3$

[2] $x(\alpha,1) - y(\alpha,1) + 13t_{4,\alpha} - 13\mathrm{Pr}_\alpha = 0$

[3] $x(\alpha,1) + z(\alpha,1) + 15t_{4,\alpha} - 15\mathrm{Pr}_\alpha = 6$

CONCLUSION

In this paper ,we have presented infinitely many non-zero distinct integer solution to the ternary quadratic equation $4x^2 - 12xy + 21y^2 = 13z^2$ representing a homogeneous cone . As diophantine equation are rich in variety , to conclude, one may search for other forms of three dimentional surfaces , namely , non-homogeneous cone , paraboloid , ellipsoid ,hyperbolic paraboloid and so on for finding integral points on them and corresponding properties

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REFERENCES

1. *L. E. Dickson, "History of Theory of Numbers and Diophantine Analysis",vol.2, Dover publications, New York 2005.*
2. *L.J. Mordell, "Diophantine Equations", Academic press, New York 1970.*
3. *R.D.Carmicheal,"The Theory of Number and Diophantine Analysis", Dover publication, New York 1959.*
4. *M.A. Gopalan and D.Geetha,Lattice Points on the Hyberboloid of two sheets,* $X^2 - 6XY + Y^2 + 6X - 2Y + 5 = z^2 + 4$ *Impact J.Sci. Tech., 4 [2012] 127-136.*
5. *M.N. Gopalan, S. Vidhyalakshimi and A.Kavitha, Integral points on the Homogeneous cone* $Z^2 = 2X^2 - 7Y^2$ *, The Diophantus J Math., 1[2] [2012] 127-136.*
6. *M.A. Gopalan, S. Vidhyalakshmi and G.Sumathi, Lattice points on the Hyperboloid one sheet* $4Z^2 = 2X^2 + 3Y^2 - 4$ *, The Diophantus J Math., 1[2] [2012] 109-115.*
7. *M.A. Gopalan, S.Vidhyalakshmi and K.Lakshmi, Integral points on the Hyperboloid two sheet* $3Y^2 = 7X^2 - Z^2 + 21$ *, The Diophantus J Math., 1[2] [2012] 99-107.*
8. *M.A. Gopalan, S.Vidhyalakshmi and S.Mallika, Observations on Hyperboloid of one sheet* $X^2 + 2Y^2 - Z^2 = 2$ *, Bessel J. Math., 2[3] [2012] 221-226.*
9. *M.A. Gopalan, S.Vidhyalakshmi, T.R Usha Rani and S.Mallika, Integral points on the Homogeneous cone* $6Z^2 + 3Y^2 - 2X^2 = 0$ *, The Impact J.Sci Tech., 6[1] [2012] 7-13.*
10. *M.A. Gopalan, S.Vidhyalakshmi and T.R.Usha Rani, Integral Points on the Non-Homogeneous cone* $2Z + 4XY + 8X - 4Z = 0$ *"Global Journal of Mathematics and Mathematical Science", 2[1] [2012] 61-67.*
11. *M.A. Gopalan, S.Vidhyalakshmi and T.R Usha Rani, Integral points on the Non- Homogeneous cone* $2Z + 4XY + 8X - 4Z = 0$ *, "Global Journal of Mathematics and Mathematical science"., 2[1] [2012] 61-67.*
12. *M.A. Gopalan, S.Vidhyalakshmi and K.Lakshmi, Lattice points on the Elliptic Paraboloid* $6Y^2 + 9Z^2 = 4X$ *, Bessel J. Math., 3[2]*
13. *M.A. Gopalan, S.Vidhyalakshmi and S.Aarthy Thangam, "On Ternary Quadratic Diophantine Equation"* $X(X + Y) = Z + 20$*, IJIRSE., 6[8] [2017] 15739-15741.*
14. *M.A. Gopalan, S.Vidhyalakshmi, E.Bhuvaneswari and R.Presenna, "On Ternary Quadratic Diophantine Equation"* $5(X + Y^2) - 6XY = 20Z^2$ *, International Journal of Advanced Scientific Research, 1[2] [2016] 59-61.*
15. *H.Ayesha Begum, T.R Usha Rani, "On Ternary Quadratic Diophantine Equation"* $3(X + Y)^2 - 2XY = 12Z^2$ *, International Research Journal of Engineering and Technology, vol:06[2019]*