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t-REGULAR t-DERIVATIONS ON p-SEMISIMPLE BCIK-ALGEBRAS

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ABSTRACT

In this paper, Introduced BCIK – algebra and its properties, and also we introduce the notion of derivation of a BCIK-algebra and investigate some related properties. We introduce the notion of t-derivation of a BCIK-algebra and investigate related properties. Moreover, we study t-derivation in a p-simisimple BCIK-algebra and establish some results on t-derivations in a p-semisimple BCIK-algebra.

KEYWORDS: ***BCIK-algebra, p-semisimple, t-derivations, t-regular.***

1. INTRODUCTION

In 1966, Y. Imai and K. Iseki [1,2] defined BCK – algebra in this notion originated from two different sources: one of them is based on the set theory the other is form the classical and non – classical propositional calculi. In [3]. Y.B. Jun and X.L. Xin applied the notion of derivation in ring and near – ring theory to BCI – algebras, and they also introduced a new concept called a derivation in BCI–algebras and its properties. We introduce combination BCK–algebra and BCI–algebra to define BCIK–algebra and its properties and also using Lattices theory to derived the some basic definitions, an algebra of type (1,0), also known as BCIK-algebra, and they also introduced a new concept called a regular derivation in BCIK-algebras. We introduce left derivation p-semisimple algebra and its properties.

After the work of Jun and Xin (2004) [3], many research articles have appeared on the derivations of BCI-algebras In different aspects as follows: in 2005 [13], Zhan and Liu have given the notion of f-derivation of BCI-algebras and studied p-semisimple BCI—algebras by using the idea of regular f-derivation in BCI-algebras. In 2006 [14] Abujabal and Al-sheshri have extended the results of BCI-algebra. Further, in the next year 2007[15] they defined and studied the notion of left derivation of BCI-algebra and incestigated some properties of left derivation in p-semisimple BCI-algebras. In 2009 [16], Ozturk and Ceven have defined the notion of derivation and generalized derivation determined by a derivation for a complicated subtraction algebra and discussed some related properties. Also, in 2009 [17], Ozturk et al. have introduced the notion of generalized derivation in BCI-algebras and established some results. Further, they have given the idea of torsion free BCI-algebra and explored some properties. In 2010 [18], Al-Shehri has applied the notion of left-right (resp.,right-left)derivation in BCI-algebra in BCI-algebra and obtained some of its properties. In 2011[19], IIbira et al, have studied the notion of left-right(resp.,right-left)symmetric biderivation in BCI-algebras.

Motivated by a lot work done on derivations of BCI-algebra and on derivations of other related abstract algebraic structures, in this paper we introduce the notion of t-derivation on BCIK-algebras and obtain some of its related properties. Further, we characterize the notion of p-semisimple BCIK-algebra xby using the notion of t-derivation and show that if d_t and d_t’ are t-derivations on X, then d_t o d_t’ is also a t-derivation and d_t o d_t’ = d_t’ o d_t. Finally, we prove that d_t’ * d_t, where d_t and d_t’ are t-derivations on a p-semisimple BCIK-algebra.

2. PRELIMINARIES

Definition 2.1 BCIK algebra

Let X be a non-empty set with a binary operation * and a constant 0. Then (X, * ,0) is called a BCIK Algebra, if it satisfies the following axioms for all x, y, z Є X:

(BCIK-1) x*y = 0, y*x = 0, z*x = 0 this imply that x = y = z.

(BCIK-2) ((x*y) * (y*z)) * (z*x) = 0.

(BCIK-3) (x*(x*y)) * y = 0.

(BCIK-4) x*x = 0, y*y = 0, z*z = 0.

(BCIK-5) 0*x =0, 0*y = 0, 0*z = 0.

For all x, y, z Є X. An inequality ≤ is a partially ordered set on X can be defined x ≤ y if and only if

(x*y) * (y*z) = 0.

Properties 2.2. [5] I any BCIK – Algebra X, the following properties hold for all x, y, z Є X:

(1) 0 Є X.
(2) x*0 = x.
(3) x*0 = 0 implies x = 0.
(4) 0*(x*y) = (0*x) * (0*y).
(5) X*y = 0 implies x = y.
(6) X*(0*y) = y*(0*x).
(7) 0*(0*x) = x.
(8) x*y Є X and x Є X imply y Є X.
(9) (x*y) * z = (x*z) * y
(10) x*(x*(x*y)) = x*y.
(11) (x*y) *(y*z) = x*y.
(12) 0 ≤ x ≤ y for all x, y Є X.
(13) x ≤ y implies x*z ≤ y*z and z*y ≤ z*x.
(14) x*y ≤ x.
(15) x*y ≤ z ⇔ x*z ≤ y for all x, y, z Є X
(16) x*a = x*b implies a = b where a and b are any natural numbers (i. e)., a, b Є N
(17) a*x = b*x implies a = b.
(18) a*(a*x) = x.

Definition 2.3. [4, 5, 6, 7] Let X be a BCIK – algebra. Then, for all x, y, z Є X:

(1) X is called a positive implicative BCIK – algebra if (x*y) * z = (x*z) * (y*z).
(2) X is called an implicative BCIK – algebra if x*(y*x) = x.
(3) X is called a commutative BCIK – algebra if x*(x*y) = y*(y*x).
(4) X is called bounded BCIK – algebra, if there exists the greatest element 1 of X, and for any x Є X, 1*x is denoted by GG_x,
(5) X is called involutory BCIK – algebra, if for all x Є X, GG_x = x.

Definition 2.4. [5, 7] Let X be a bounded BCIK-algebra. Then for all x, y Є X:

(1) $G1 = 0$ and $G0 = 1$,
(2) $GG_x \leq x$ that $GG_x = G(G_x)$,
(3) $G_x * G_y \leq y*x$,
(4) $y \leq x$ implies $G_x \leq G_y$,
(5) $G_{x*y} = G_{y*x}$
(6) $GGG_x = G_x$.

Theorem 2.5. [8] Let X be a bounded BCIK-algebra. Then for any x, y Є X, the following hold:

(1) X is involutory,
(2) $x*y = G_y * G_x$,
(3) $x*G_y = y * G_x$,
(4) $x \leq G_y$ implies $y \leq G_x$.

Theorem 2.6. [5] Every implicative BCIK-algebra is a commutative and positive implicative BCIK-algebra.

Definition 2.7. [10,11] Let X be a BCIK-algebra. Then:

(1) X is said to have bounded commutative, if for any x, y Є X, the set A(x,y) = {t Є X : $t*x \leq y$} has the greatest element which is denoted by x o y,
(2) (X, * , ≤) is called a BCIK-lattices, if (X,≤) is a lattice, where ≤ is the partial BCIK-order on X, which has been introduced in Definition 2.1.

Definition 2.8. [11] Let X be a BCIK-algebra with bounded commutative. Then for all x, y, z Є X:

(1) $y \leq x \text{ o } (y*x)$,
(2) $(x \text{ o } z) * (y \text{ o } z) \leq x*y$,
(3) $(x*y) * z = x*(y \text{ o } z)$,
(4) If $x \leq y$, then $x \text{ o } z \leq y \text{ o } z$,
(5) $z*x \leq y \Leftrightarrow z \leq x \text{ o } y$.

Theorem 2.9. [12] Let X be a BCIK-algebra with condition bounded commutative. Then, for all x, y, z Є X, the following are equivalent:

(1) X is a positive implicative,
(2) $x \leq y$ implies $x \text{ o } y = y$,
(3) $x \text{ o } x = x$,
(4) $(x \text{ o } y) * z = (x*z) \text{ o } (y*z)$,
(5) $x \text{ o } y = x \text{ o } (y*x)$.

Theorem 2.10. [8, 9, 10] Let X be a BCIK-algebra.

(1) If X is a finite positive implicative BCIK-algebra with bounded and commutative the (X, ≤) is a distributive lattice,
(2) If X is a BCIK-algebra with bounded and commutative, then X is positive implicative if and only if (X, ≤) is an upper semi lattice with $x \vee y = x \text{ o } y$, for any x, y Є X,
(3) If X is bounded commutative BCIK-algebra, then BCIK-lattice (X, ≤) is a distributive lattice, where $x \wedge y = y*(y*x)$ and $x \vee y= G(G_x \wedge G_y)$.

Theorem 2.11. [8] Let X be an involutory BCIK-algebra, Then the following are equivalent:

(1) (X, ≤) is a lower semi lattice,

(2) (X, ≤) is an upper semi lattice,
(3) (X, ≤) is a lattice.

Theorem 2.12. [6] Let X be a bounded BCIK-algebra. Then:

(1) every commutative BCIK-algebra is an involutory BCIK-algebra.
(2) Any implicative BCIK-algebra is a Boolean lattice (a complemented distributive lattice).

Theorem 2.13. [7, 9] Let X be a BCK-algebra, Then, for all x, y, z Є X, the following are equivalent:
(1) X is commutative,
(2) x*y = x*(y*(y*x)),
(3) x*(x*y) = y*(y*(x*(x*y))),
(4) x≤ y implics x = y*(y*x).

3. Regular Left derivation p-semisimple BCIK-algebra

Definition 3.1. Let X be a p-semisimple BCIK-algebra. We define addition + as x + y = x*(0*y) for all x, y Є X. Then (X,+) be an abelian group with identity 0 and x –y = x*y. Conversely, let (X,+) be an abelian group with identity 0 and let x – y =x*y. Then X is a p-semisimple BCIK-algebra and x + y = x*(0*y), for all x, y Є X (see [16]). We denote x □ y = y * (y * x), 0 * (0 * x) = a_x and
L_p(X) = {a Є X / x * a = 0 implies x = a, for all x Є X}.
For any x Є X. V(a) = {a Є X / x * a = 0} is called the branch of X with respect to a. We have x * y Є V(a * b), whenever x Є V(a) and y Є V(b), for all x, y Є X and all a, b Є L_p(X), for 0 * (0 * a_x) = a_x which implies that a_x * y Є L_p(X) for all y Є X. It is clear that G(X) ⊂ L_p(X) and x * (x * a) = a and a * x Є L_p(X), for all a Є L_p(X) and all x Є X. For more detail, we refer to [17,18,19,20,21].

Definition 3.2. ([3]) Let X be a BCIK-algebra. By a (l, r)-derivation of X, we mean a self d of X satisfying the identity

d(x * y) = (d(x) * y) ∧ (x * d(y)) for all x, y Є X.

If X satisfies the identity

d(x * y) = (x * d(y)) ∧ (d(x) * y) for all x, y Є X,

then we say that d is a (r, l)-derivation of X
Moreover, if d is both a (r, l)-derivation and (r, l)-derivation of X, we say that d is a derivation of X.

Definition 3.3. ([3]) A self-map d of a BCIK-algebra X is said to be regular if d(0) = 0.

Definition 3.4. ([3]) Let d be a self-map of a BCIK-algebra X. An ideal A of X is said to be d-invariant, if d(A) = A.
In this section, we define the left derivations

Definition 3.5. Let X be a BCIK-algebra By a left derivation of X, we mean a self-map D of X satisfying

D(x * y) = (x * D(y)) ∧ (y * D(x)), for all x, y Є X.

Example 3.6. Let X = {0,1,2} be a BCIK-algebra with Cayley table defined by

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a map D: X → X by

$$D(x) = \begin{cases} 2 \text{ if } x = 0,1 \\ 0 \text{ if } x = 2. \end{cases}$$

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Then it is easily checked that D is a left derivation of X.

Proposition 3.7. Let D be a left derivation of a BCIK-algebra X. Then for all x, y Є X, we have

(1) $x * D(x) = y * D(y)$.
(2) $D(x) = a_{D(x)\square x}$.
(3) $D(x) = D(x) \wedge x$.
(4) $D(x)$ Є $L_p(X)$.

Proof.

(1) Let x, y Є X. Then

$$D(0) = D(x * x) = (x * D(x)) \wedge (x * D(x)) = x * D(x).$$

Similarly, $D(0) = y * D(y)$. So, $D(x) = y * D(y)$.

2) Let x Є X. Then

$$\begin{aligned} D(x) &= D(x * 0) \\ &= (x * D(0)) \wedge (0 * D(x)) \\ &= (0 * D(x)) * ((0 * D(x)) * (x * D(0))) \\ &\leq 0 * (0 * (x * D(x)))) \\ &= 0 * (0 * (x * (x * D(x)))) \\ &= 0 * (0 * (D(x) \wedge x)) \\ &= a_{D(x)\square x}. \end{aligned}$$

Thus $D(x) \leq a_{D(x)\square x}$. But

$$a_{D(x)\square x} = 0(0 * (D(x) \wedge x)) \leq D(x) \wedge x \leq D(x).$$

Therefore, $D(x) = a_{D(x)\square x}$.

(3) Let x Є X. Then using (2), we have

$$D(x) = a_{D(x)\square x} \leq D(x) \wedge x .$$

But we know that $D(x) \wedge x \leq D(x)$, and hence (3) holds.

(4) Since a_x Є $L_p(X)$, for all x Є X, we get $D(x)$ Є $L_p(X)$ by (2).

Remark 3.8. Proposition 3.3(4) implies that D(X) is a subset of $L_p(X)$.

Proposition 3.9. Let D be a left derivation of a BCIK-algebra X. Then for all x, y Є X, we have

(1) $Y * (y * D(x)) = D(x)$.
(2) $D(x) * y$ Є $L_p(X)$.

Proposition 3.10. Let D be a left derivation of a BCIK-algebra of a BCIK-algebra X. Then

(1) $D(0)$ Є $L_p(X)$.
(2) $D(x) = 0 + D(x)$, for all x Є X.
(3) $D(x + y) = x + D(y)$, for all x, y Є $L_p(X)$.
(4) $D(x) = x$, for all x Є X if and only if $D(0) = 0$.
(5) $D(x)$ Є G(X), for all x Є G(X).

Proof.

(1) Follows by Proposition 3.3(4).
(2) Let x Є X. From Proposition 3.3(4), we get $D(x) = a_{D(x)}$, so we have

$$D(x) = a_{D(x)} = 0 * (0 * D(x)) = 0 + D(x).$$

(3) Let x, y Є $L_p(X)$. Then

$$\begin{aligned} D(x + y) &= D(x * (0 * y)) \\ &= (x * D(0 * y)) \wedge ((0 * y) * D(x)) \\ &= ((0 * y) * D(x)) * (((0 * y) * D(x) * (x * D(0 * y))) \end{aligned}$$

= x * D(0 * y)
= x * ((0 * D(y)) ∧ (y * D(0)))
= x * D(0 *y)
= x * (0 * D(y))
= x + D(y).

(4) Let D(0) = 0 and x Є X. Then
D(x) = D(x) ∧ x = x * (x * D(x)) = x * D(0) = x * 0 = x.
Conversely, let D(x) = x, for all x Є X. So it is clear that D(0) = 0.

(5) Let x Є G(x). Then 0 * = x and so
D(x) = D(0 * x)
= (0 * D(x)) ∧ (x * D(0))
= (x * D(0)) * ((x * D(0)) * (0 * D(x))
= 0 * D(x).
This give D(x) Є G(X).

Remark 3.11. Proposition 3.6(4) shows that a regular left derivation of a BCIK-algebra is the identity map. So we have the following:

Proposition 3.12. A regular left derivation of a BCIK-algebra is trivial.

Remark 3.13. Proposition 3.6(5) gives that D(x) Є G(X) ⊆ L_p(X).

Definition 3.14. An ideal A of a BCIK-algebra X is said to be D-invariant if D(A) ⊂ A.
Now, Proposition 3.8 helps to prove the following theorem.

Theorem 3.15. Let D be a left derivation of a BCIK-algebra X. Then D is regular if and only if ideal of X is D-invariant.

Proof.

Let D be a regular left derivation of a BCIK-algebra X. Then Proposition 3.8. gives that D(x) = x, for all x Є X. Let y Є D(A), where A is an ideal of X. Then y = D(x) for some x Є A. Thus

Y * x = D(x) * x = x * x = 0 Є A.

Then y Є A and D(A) ⊂ A. Therefore, A is D-invarient.

Conversely, let every ideal of X be D-invariat. Then D({0}) ⊂ {0} and hence D(0) and D is regular.
Finally, we give a characterization of a left derivation of a p-semisimple BCIK-algebra.

Proposition 3.16. Let D be a left derivation of a p-semisimple BCIK-algebra. Then the following hold for all x, y Є X:

(1) D(x * y) = x * D(y).
(2) D(x) * x = D(y) * Y.
(3) D(x) * x = y * D(y).

Proof.

(1) Let x, y Є X. Then
D(x * y) = (x * D(y)) ∧ ∧ (y * D(x)) = x * D(y).

(2) We know that
(x * y) * (x * D(y)) ≤ D(y) * y and
(y * x) * (y * D(x)) ≤ D(x) * x.
This means that
((x * y) * (x * D(y))) * (D(y) * y) = 0, and
((y * x) * (y * D(x))) * (D(x) * x) = 0.
So
((x * y) * (x * D(y))) * (D(y) * y) = ((y * x) * (y * D(x))) * (D(x) * x). (I)
Using Proposition 3.3(1), we get,
(x * y) * D(x * y) = (y * x) * D(y * x). (II)

By (I), (II) yields
$(x * y) * (x * D(y)) = (y * x) * (y * D(x))$.
Since X is a p-semisimple BCIK-algebra. (I) implies that
$D(x) * x = D(y) * y$.

(3) We have, $D(0) = x * D(x)$. From (2), we get $D(0) * 0 = D(y) * y$ or $D(0) = D(y) * y$.
So $D(x) * x = y * D(y)$.

Theorem 3.17. In a p-semisimple BCIK-algebra X a self-map D of X is left derivation if and only if and if it is derivation.

Proof.

Assume that D is a left derivation of a BCIK-algebra X. First, we show that D is a (r,l)-derivation of X. Then

$$\begin{aligned} D(x * y) &= x * D(y) \\ &= (D(x) * y) * ((D(x) * Y) * (x * D(y))) \\ &= (x * D(y)) \wedge (D(x) * y). \end{aligned}$$

Now, we show that D is a (r,l)-derivation of X. Then

$$\begin{aligned} D(x * Y) &= x * D(y) \\ &= (x * 0) * D(y) \\ &= (x * (D(0) * D(0)) * D(y) \\ &= (x * ((x * D(x)) * (D(y) * y))) * D(y) \\ &= (x * ((x * D(y)) * (D(x) * y))) * D(y) \\ &= (x * D(y) * ((x * D(y)) * (D(x) * Y)) \\ &= (D(x) * y) \wedge (x * D(y)). \end{aligned}$$

Therefore, D is a derivation of X.

Conversely, let D be a derivation of X. So it is a (r,l)-derivation of X. Then

$$\begin{aligned} D(x * y) &= (x * D(y)) \wedge (D(x) * y) \\ &= (D(x) * y) * ((D(x) * y) * (x * D(y))) \\ &= x * D(y) = (y * D(x)) * ((y * D(x)) * (x * D(y))) \\ &= (x * D(y)) \wedge (y * D(x)). \end{aligned}$$

Hence, D is a left derivation of X.

4. t-Derivations in a BCIK-algebra/p-Semisimple BCIK-algebra

The following definitions introduce the notion of t-derivation for a BCIK-algebra.

Definition 4.1. Let X be a BCIK-algebra. Then for $t \in X$, we define a self map $d_t : X \to X$ by $d_t(x) = x * t$ for all $x \in X$.

Definition 4.2. Let X be a BCIK-algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a left-rifht t-derivation or (l,r)-t-derivation of X if it satisfies the identity $d_t(x * Y) = (d_t(x) * y) \wedge (x * d_t(y))$ for all $x, y \in X$.

Definition 4.3. Let X be a BCIK-algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a left-right t-derivation or (l,r)-t-derivation of X if it satisfies the identity $d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$ for all $x, y \in X$.
Moreover, if d_t is both a (l,r) and a(r.l)-t-derivation on X, we say that d_t is a t-derivation on X.

Example 4.4. Let X = {0,1,2} be a BCIK-algebra with the following Cayley table:

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

For any $t \in X$, define a self map $d_t : X \to X$ by $d_t(x) = x * t$ for all $x \in X$. Then it is easily checked that d_t is a t-derivation of X.

Proposition 4.5. Let d_t be a self map of an associative BCIK-algebra X. Then d_t is a (l,r)-t-derivation of X.
Proof. Let X be an associative BCIK-algebra, then we have

$$d_t(x * y) = (x * y)$$

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$= \{x * (y * t)\} * 0$
$= \{x * (y * t)\} * [\{x * (y * t)\} * \{x * (y * t)\}]$
$= \{x * (y * t)\} * [\{x * (y * t)\} * \{(x * y) * t\}]$
$= \{x * (y * t)\} * [\{x * (y * t)\} * \{(x * t) * y\}]$
$= ((x * t) * y) \wedge (x * (y * t))$
$= (d_t(x) * y) \wedge (x * d_t(y))$.

Proposition 4.6. Let d_t be a self map of an associative BCIK-algebra X. Then, d_t is a (r,l)-t-derivation of X.
Proof. Let X be an associative BCIK-algebra, then we have

$d_t(x * y) = (x * y) * t$
$= \{(x * t) * y\} * 0$
$= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * t) * y)]$
$= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * y) * t\}]$
$= \{(x * t) * y\} * [\{(x * t) * y\} * \{x * (y * t)\}]$
$= (x * (y * t)) \wedge ((x * t) * y)$
$= (x * d_t(y)) \wedge (d_t(x) * y)$

Combining Propositions 4.5 and 4.6, we get the following Theorem.

Theorem 4.7. Let d_t be a self map of an associative BCIK-algebra X. Then, d_t is a t-derivation of x.

Definition 4.8. A self map d_t of a BCIK-algebra X is said to be t-regular if $d_t(0) = 0$.

Example 4.9. Let X = {0, a, b} be a BCIK-algebra with the following Cayley table:

*	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(1) For any t Є X, define a self map $d_t : X \rightarrow X$ by

$$d_t(x) = x * t = \begin{cases} b \text{ if } x = 0, a \\ 0 \text{ if } x = b \end{cases}$$

Then it is easily checked that d_t is (l,r) and (r,l)-t-derivations of X, which is not t-regular.

(2) For any t Є X, define a self map $d'_t : X \rightarrow X$ by

$$d_t'(x) = x * t = \begin{matrix} 0 \text{ if } x = 0, a \\ b \text{ if } x = b. \end{matrix}$$

Then it is easily checked that d_t' is (l,r) and (r,l)-t-derivations of X, which is t-regular.

Proposition 4.10. Let d_t be a self map of a BCIK-algebra X. Then

(1) If d_t is a (l,r)-t- derivation of x, then $d_t(x) = d_t(x) \wedge x$ for all x Є X.
(2) If d_t is a (r,l)-t-derivation of X, then $d_t(x) = x \wedge d_t(x)$ for all x Є X if and only if d_t is t-regular.

Proof.

(1) Let d_t be a (l,r)-t-derivation of X, then

$d_t(x) = d_t(x * 0)$
$= (d_t(x) * 0) \wedge (x * d_t(0))$
$= d_t(x) \wedge (x * d_t(0))$
$= \{x * d_t(0)\} * [\{x * d_t(0)\} * d_t(x)]$
$= \{x * d_t(0)\} * [\{x * d_t(x)\} * d_t(0)]$
$\leq x * \{x * d_t(x)\}$
$= d_t(x) \wedge x$.

But $d_t(x) \wedge x \leq d_t(x)$ is trivial so (1) holds.

(2) Let d_t be a (r,l)-t-derivation of X. If $d_t(x) = x \leq d_t(x)$ then

$d_t(0) = 0 \wedge d_t(0)$

$= d_t(0) * \{ d_t(0) * 0\}$
$= d_t(0) * d_t(0)$
$= 0$

Thereby implying d_t is t-regular. Conversely, suppose that d_t is t-regular, that is $d_t(0) = 0$, then we have

$$\begin{aligned} d_t(0) &= d_t(x * 0) \\ &= (x * d_t(0)) \wedge (d_t(x) * 0) \\ &= (x * 0) \wedge d_t(x) \\ &= x \wedge d_t(x). \end{aligned}$$

The completes the proof.

Theorem 4.11. Let d_t be a (l,r)-t-derivation of a p-semisimple BCIK-algebra X. Then the following hold:

(1) $d_t(0) = d_t(x) * x$ for all $x \in X$.
(2) d_t is one-0ne.
(3) If there is an element $x \in X$ such that $d_t(x) = x$, then d_t is identity map.
(4) If $x \le y$, then $d_t(x) \le d_t(y)$ for all $x, y \in X$.

Proof.

(1) Let d_t be a (l,r)-t-derivation of a p-semisimple BCIK-algebra X. Then for all $x \in X$, we have $x * x = 0$ and so

$$\begin{aligned} d_t(0) &= d_t(x * x) \\ &= (d_t(x) * x) \wedge (x * d_t(x)) \\ &= \{x * d_t(x)\} * [\{x * d_t(x)\} * \{d_t(x) * x\}] \\ &= d_t(x) * x \end{aligned}$$

(2) Let $d_t(x) = d_t(y) \Rightarrow x * t = y * t$, then we have $x = y$ and so d_t is one-one.
(3) Let d_t be t-regular and $x \in X$. Then, $0 = d_t(0)$ so by the above part(1), we have $0 = d_t(x) * x$ and, we obtain $d_t(x) = x$ for all $x \in X$. Therefore, d_t is the identity map.
(4) It is trivial and follows from the above part (3).
(5) Let $x \le y$ implying $x * y = 0$. Now,

$$\begin{aligned} d_t(x) * d_t(y) &= (x * t) * (y * t) \\ &= x * y \\ &= 0. \end{aligned}$$

Therefore, $d_t(x) \le d_t(y)$. This completes proof.

Definition 4.12. Let d_t be a t-derivation of a BCIK-algebra X. Then, d_t is said to be an isotone t-derivation if $x \le y \Rightarrow d_t(x) \le d_t(y)$ for all $x, y \in X$.

Example 4.13. In Example 4.9(2), d_t' is an isotone t-derivation, while in Example 4.9(1), d_t is not an isotone t-derivation.

Proposition 4.14. Let X be a BCIK-algebra and d_t be a t-derivation on X. Then for all $x, y \in X$, the following hold:

(1) If $d_t(x \wedge y) = d_t(x)\, d_t(x)\, d_t(x)$, then d_t is an isotone t-derivation
(2) If $d_t(x \wedge y) = d_t(x) * d_t(y)$, then d_t is an isotone t-derivation.

Proof.

(1) Let $d_t(x \wedge y) = d_t(x) \wedge d_t(x)$. If $x \le y \Rightarrow x \wedge y = x$ for all $x, y \in X$. Therefore, we have

$$\begin{aligned} d_t(x) &= d_t(x \wedge y) \\ &= d_t(x) \wedge d_t(y) \\ &\le d_t(y). \end{aligned}$$

Henceforth $d_t(x) \le d_t(y)$ which implies that d_t is an isotone t-derivation.

(2) Let $d_t(x * y) = d_t(x) * d_t(y)$. If $x \le y \Rightarrow x * y = 0$ for all $x, y \in X$. Therefore, we have

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= d_t\{x * (x * y)\} \\ &= d_t(x) * d_t(x * y) \\ &= d_t(x) * \{ d_t(x) * d_t(y)\} \\ &\le d_t(y). \end{aligned}$$

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Thus, $d_t(x) \leq d_t(y)$. This completes the proof.

Theorem 4.15. Let d_t be a t-regular (r,l)-t-derivation of a BCIK-algebra X. Then, the following hold:

(1) $d_t(x) \leq x$ for all $x \in X$.
(2) $d_t(x) * y \leq x * d_t(y)$ for all $x, y \in X$.
(3) $d_t(x * y) = d_t(x) * y \leq d_t(x) * d_t(y)$ for all $x, y \in X$.
(4) $Ker(d_t) := \{ x \in X : d_t(x) = 0\}$ is a subalgebra of X.

Proof.

(1) For any $x \in X$, we have $d_t(x) = d_t(x * 0) = (x * d_t(0)) \wedge (d_t(x) * 0) = (x * 0) \wedge (d_t(x) * 0) = x \wedge d_t(x) \leq x$.
(2) Since $d_t(x) \leq x$ for all $x \in X$, then $d_t(x) * y \leq x * y \leq x * d_t(y)$ and hence the proof follows.
(3) For any $x, y \in X$, we have

$$\begin{aligned} d_t(x * y) &= (x * d_t(y)) \wedge (d_t(x) * y) \\ &= \{d_t(x) * y\} * [\{d_t(x) * y\} * \{x * d_t(x)\}] \\ &= \{d_t(x) * y\} * 0 \\ &= d_t(x) * y \leq d_t(x) * d_t(x). \end{aligned}$$

(4) Let $x, y \in ker(d_t) \Rightarrow d_t(x) = 0 = d_t(y)$. From (3), we have $d_t(x * y) \leq d_t(x) * d_t(y) = 0 * 0 = 0$ implying $d_t(x * y) \leq 0$ and so $d_t(x * y) = 0$. Therefore, $x * y \in ker(d_t)$. Consequently $ker(d_t)$ is a subalgebra of X. This completes the proof.

Definition 4.16. Let X be a BCIK-algebra and let d_t, d_t' be two self maps of X. Then we define $d_t \circ d_t' : X \rightarrow X$ by $(d_t \circ d_t')(x) = d_t(d_t'(x))$ for all $x \in X$.

Example 4.17. Let X = {0, a, b} be a BCIK-algebra which is given in Example 4.4. Let d_t and d_t' be two self maps on X as define in Example 4.9(1) and Example 4.9(2), respectively.

Now, define a self map $d_t \circ d_t' : X \rightarrow X$ by

$$(d_t \circ d_t')(x) = \begin{cases} 0 \; if \;\; x = a, b \\ b \; if \;\; x = 0. \end{cases}$$

Then, it easily checked that $(d_t \circ d_t')(x) = d_t(d_t'(x))$ for all $x \in X$.

Proposition 4.18. Let X be a p-semisimple BCIK-algebra X and let d_t, d_t' be (l,r)-t-derivations of X. Then, $d_t \circ d_t'$ is also a (l,r)-t-derivation of X.

Proof. Let X be a p-semisimple BCIK-algebra. d_t and d_t' are (l,r)-t-derivations of X. Then for all $x, y \in X$, we get

$$\begin{aligned} (d_t \circ d_t')(x * y) &= d_t(d_t'(x,y)) \\ &= d_t[(d_t'(x) * y) \wedge (x * d_t(y))] \\ &= d_t[(x * d_t'(y)) * \{(x * d_t(y)) * (d_t'(x) * y)\}] \\ &= d_t(d_t'(x) * y) \\ &= \{x * d_t(d_t'(y))\} * [\{x * d_t(d_t'(y))\} * \{d_t(d_t'(x) * y)\}] \\ &= \{ d_t(d_t'(x) * y)\} \wedge \{x * d_t(d_t'(y))\} \\ &= ((d_t \circ d_t')(x) * y) \wedge (x * (d_t \circ d_t')(y)). \end{aligned}$$

Therefore, $(d_t \circ d_t')$ is a (l,r)-t-derivation of X.

Similarly, we can prove the following.

Proposition 4.19. Let X be a p-semisimple BCIK-algebra and let d_t, d_t' be (r,l)-t-derivations of X. Then, $d_t \circ d_t'$ is also a (r,l)-t-derivation of X.

Combining Propositions 3.18 and 3.19, we get the following.

Theorem 4.20. Let X be a p-semisimple BCIK-algebra and let d_t, d_t' be t-derivations of X. Then, $d_t \circ d_t'$ is also a t-derivation of X.

Now, we prove the following theorem

Theorem 4.21. Let X be a p-semisimple BCIK-algebra and let d_t, d_t' be t-derivations of X. Then $d_t \circ d_t' = d_t' \circ d_t$.

Proof. Let X be a p-semisimple BCIK-algebra. d_t and d_t', t-derivations of X. Suppose d_t' is a (l,r)-t-derivation, then for all $x, y \in X$, we have

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$$
\begin{aligned}
(d_t \circ d_t')(x * y) &= d_t(d_t'(x * y)) \\
&= d_t[(d_t'(x) * y) \wedge (x * d_t(y))] \\
&= d_t[(x * d_t'(y)) * \{(x * d_t(y)) * (d_t'(x) * y)\}] \\
&= d_t(d_t'(x) * y)
\end{aligned}
$$

As d_t is a (r,l)-t-derivation, then

$$
\begin{aligned}
&= (d_t'(x) * d_t(y)) \wedge (d_t(d_t'(x)) * y) \\
&= d_t'(x) * d_t(y).
\end{aligned}
$$

Again, if d_t is a (r,l)-t-derivation, then we have

$$
\begin{aligned}
(d_t \circ d_t')(x * y) &= d_t'[d_t(x * y)] \\
&= d_t'[(x * d_t(y)) \wedge (d_t(x) * y)] \\
&= d_t'[x * d_t(y)]
\end{aligned}
$$

But d_t' is a (l,r)-t-derivation, then

$$
\begin{aligned}
&= (d_t'(x) * d_t(y)) \wedge (x * d_t'(d_t(y)) \\
&= d_t'(x) * d_t(y)
\end{aligned}
$$

Therefore, we obtain

$$(d_t \circ d_t')(x * y) = (d_t' \circ d_t)(x * y).$$

By putting y = 0, we get

$$(d_t \circ d_t')(x) = (d_t' \circ d_t)(x) \text{ for all } x \in X.$$

Hence, $d_t \circ d_t' = d_t' \circ d_t$. This completes the proof.

Definition 4.22. Let X be a BCIK-algebra and let d_t, d_t' two self maps of X. Then we define $d_t * d_t' : X \to X$ by $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all $x \in X$.

Example 4.23. Let X = {0, a, b} be a BCIK-algebra which is given in Example 3.4. let d_t and d_t' be two self maps on X as defined in Example 4.9 (1) and Example 4.10 (2), respectively.

Now, define a self map $d_t * d_t' : X \to X$ by

$$(d_t * d_t')(x) = \begin{cases} 0 \text{ if } x = a, b \\ b \text{ if } x = 0. \end{cases}$$

Then, it is easily checked that $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all $x \in X$.

Theorem 4.24. Let X be a p-semisimple BCIK-algebra and let d_t, d_t' be t-derivations of X. Then $d_t * d_t' = d_t' * d_t$.
Proof. Let X be a p-semisimple BCIK-algebra. d_t and d_t', t-derivations of X.
Since d_t' is a (r,l)-t-derivation of X, then for all x, y $\in$ X, we have

$$
\begin{aligned}
(d_t \circ d_t')(x * y) &= d_t(d_t'(x * y)) \\
&= d_t[(x * d_t'(y)) \wedge (d_t'(x) * y)] \\
&= d_t[(x * d_t'(y)]
\end{aligned}
$$

But d_t is a (l,r)-r-derivation, so

$$
\begin{aligned}
&= (d_t(x) * d_t'(y)) \wedge (x * d_t(d_t'(y)) \\
&= d_t(x) * d_t'(x).
\end{aligned}
$$

Again , if d_t' is a (l,r)-t-derivation of X, then for all x, y $\in$ X, we have

$$
\begin{aligned}
(d_t \circ d_t')(x * y) &= d_t[d_t'(x * y)] \\
&= d_t[(d_t'(x) * y) \wedge (x * d_t'(y))] \\
&= d_t[(x * d_t'(y)) * \{(x * d_t'(y)) * (d_t'(x) * y)\}] \\
&= d_t(d_t'(x) * y).
\end{aligned}
$$

As d_t is a (r,l)-t-derivation, then

$$
\begin{aligned}
&= (d_t'(x) * d_t(y)) \wedge (d_t(d_t'(x)) * y) \\
&= d_t'(x) * d_t(y).
\end{aligned}
$$

Henceforth , we conclude

$$d_t(x) * d_t'(y) = d_t'(x) * d_t(y)$$

By putting y =x, we get

$$d_t(x) * d_t'(x) = d_t'(x) * d_t(x)$$
$$(d_t * d_t')(x) = (d_t' * d_t)(x) \text{ for all } x \in X.$$

Hence $d_t * d_t' = d_t' * d_t$. This completes the proof.

5. CONCLUSION

Derivation is a very interesting and are of research in the theory of algebraic structures in mathematics. The theory of derivations of algebraic structures is a direct descendant of the development of classical Galosis theory. In this paper, we have considered the notation of t-derivations in BCIK-algebra and investigated the useful properties of the t-derivations in BCIK-algebra. Finally, we investigated the notion of t-derivations in a p-semisimple BCIK-algebra and established some results on t-derivations in a p-semisimple BCIK-algebra. In our opinion, these definitions and main results can be similarly extended to some other algebraic system such as subtraction algebras, B-algebras, MV-algebras, d-algebras, Q-algebras and so forth.

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