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A SEARCH ON THE INTEGER SOLUTIONS TO TERNARY QUADRATIC DIOPHANTINE EQUATION $z^2 = Dx^2 + y^2, D = \text{odd prime}$

K.Meena [1], S.Vidhyalakshmi[2] , M.A. Gopalan[3]

[1] *Former VC, Bharathidasan University, Trichy-620 002, Tamil Nadu, India.*

[2] *Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University,Trichy-620 002,Tamil Nadu, India.*

[3] *Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002,Tamil Nadu, India.*

ABSTRACT

The homogeneous ternary quadratic diophantine equation given by $z^2 = Dx^2 + y^2, D = odd\ prime$ ***is analyzed for its non-zero distinct integer solutions through different methods. Also, formulae for generating sequence of integer solutions based on the given solutions are presented.***

KEYWORDS: ***Ternary quadratic, Integer solutions, Homogeneous cone.***

INTRODUCTION

It is well known that the quadratic diophantine equations with three unknowns (homogenous (or) non-homogenous) are rich in variety [1,2]. In particular, the ternary quadratic diophantine equations of the form $z^2 = Dx^2 + y^2$ are analyzed for values of $D = 29,41,43,47,55,61,63,67$ in [3-10]. These results motivated us to obtain non-zero distinct integer solutions to the homogeneous ternary quadratic diophantine equation given by $z^2 = Dx^2 + y^2, D = odd\ prime$ through different methods. Also, formulas for generating sequence of integer solutions based on the given solutions are presented.

METHOD OF ANALYSIS

The ternary quadratic diophantine equation to be solved for its integer solutions is

$$z^2 = Dx^2 + y^2, D = odd\ prime \qquad (1)$$

We present below different methods of solving (1)

Method: 1

(1) is written in the form of ratio as

$$\frac{z+y}{x} = \frac{Dx}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (2)$$

which is equivalent to the system of double equations

$$\alpha x - \beta y - \beta z = 0$$
$$Dx\beta + \alpha y - \alpha z = 0$$

Applying the method of cross-multiplication to the above system of equations, one obtains

$$x = x(\alpha, \beta) = 2\alpha\beta$$
$$y = y(\alpha, \beta) = \alpha^2 - D\beta^2$$
$$z = z(\alpha, \beta) = \alpha^2 + D\beta^2$$

which satisfy (1).

Note: 1

It is observed that (1) may also be represented as below:

$$\frac{z+y}{Dx} = \frac{x}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0$$

Employing the procedure as above , the corresponding solutions to (1) are given by :

$$x = 2\alpha\beta,\ y = D\alpha^2 - \beta^2,\ z = D\alpha^2 + \beta^2$$

Method: 2

(1) is written as the system of double equations in Table 1 as follows:

Table: 1 System of Double Equations

System	I	II	III
$z+y=$	Dx	x^2	Dx^2
$z-y=$	x	D	1

Solving each of the above system of double equations, the values of x, y & z satisfying (1) are obtained. For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: I

$$x = k,\ \ y = \frac{(D-1)}{2}k\ ,\ z = \frac{(D+1)}{2}k$$

Solutions for system: II

$$x = 2k+1, \quad y = 2k^2 + 2k - \frac{(D-1)}{2} \quad z = 2k^2 + 2k + \frac{(D+1)}{2}$$

Solutions for system: III

$$x = 2k+1, \; y = D(2k^2 + 2k) + \frac{(D-1)}{2}, \; z = D(2k^2 + 2k) + \frac{(D+1)}{2},$$

Method: 3

Let $z = y + k, \qquad k \neq 0$ (3)

$$\therefore (1) \Rightarrow 2ky = Dx^2 - k^2$$

Assume

$$x = k\left(2\alpha + 1\right) \tag{4}$$

$$\therefore y = D\left(2k\alpha^2 + 2k\alpha\right) + \frac{(D-1)}{2}k \tag{5}$$

In view of (3),

$$z = D\left(2k\alpha^2 + 2k\alpha\right) + \frac{(D+1)}{2}k \tag{6}$$

Note that (4), (5), (6) satisfy (1).

Method: 4

(1) is written as

$$y^2 + Dx^2 = z^2 = z^2 * 1 \tag{7}$$

Assume z as

$$z = a^2 + Db^2 \tag{8}$$

Write 1 as

$$1 = \frac{\left[D - 1 + i\,2\sqrt{D}\right]\left[D - 1 - i\,2\sqrt{D}\right]}{(D+1)^2} \tag{9}$$

Using (8) & (9) in (7) and employing the method of factorization, consider

$$\left(y + i\sqrt{D}\,x\right) = \left(a + i\sqrt{D}\,b\right)^2 . \frac{\left[D - 1 + i\,2\sqrt{D}\right]}{\left(D+1\right)}.$$

Equating the real&imaginary parts, it is seen that

$$\left.\begin{aligned} x &= \frac{1}{(D+1)}\left[2(D-1)ab + 2\left[a^2 - Db^2\right]\right] \\ y &= \frac{1}{(D+1)}\left[(D-1)\left[a^2 - Db^2\right] - 4Dab\right] \end{aligned}\right\} \quad (10)$$

Since our interest is to find the integer solutions, replacing a by $(D+1)A$ & b by $(D+1)B$ in (10) & (8) , the corresponding integer solutions to (1) are given by

$$\begin{aligned} x &= x(A,B) = (D+1)\left[2(D-1)AB + 2\left[A^2 - DB^2\right]\right], \\ y &= y(A,B) = (D+1)\left[(D-1)\left[A^2 - DB^2\right] - 4DAB\right], \\ z &= z(A,B) = (D+1)^2\left[A^2 + DB^2\right] \end{aligned}$$

Note 2:

It is worth to observe that, one may write 1 as follows:

$$1 = \frac{\left[(Dr^2 - s^2) + i\sqrt{D}\cdot 2rs\right]\left[(Dr^2 - s^2) - i\sqrt{D}\cdot 2rs\right]}{(Dr^2 + s^2)^2}$$

$$1 = \frac{\left[\left(2k^2 + 2k - \frac{(D-1)}{2}\right) + i\sqrt{D}(\cdot 2k + 1\right]\left[\left(2k^2 + 2k - \frac{(D-1)}{2}\right) - i\sqrt{D}\cdot(2k+1)\right]}{\left(2k^2 + 2k + \frac{(D+1)}{2}\right)^2}$$

$$1 = \frac{\left[(2k^2 + 2k)D + \frac{(D-1)}{2} + i\sqrt{D}(\cdot 2k + 1\right]\left[(2k^2 + 2k)D + \frac{(D-1)}{2} - i\sqrt{D}\cdot(2k+1)\right]}{\left|(2k^2 + 2k)D + \frac{(D+1)}{2}\right|^2}$$

Following the above procedure,one may obtain difference sets of integer solutions to (1).

Method 5:

(1) is also written as

$$z^2 - Dx^2 = y^2 = y^2 * 1$$

Assume y as

$$y = a^2 - Db^2$$

Note that 1 may be represented as follows:

Choice (i) : $1=\frac{\left(D+1+2\sqrt{D}\right)\left(D+1-2\sqrt{D}\right)}{(D-1)^2}$

Choice (ii): $1=\frac{\left[\left(Dr^2+s^2\right)+\sqrt{D}\cdot 2rs\right]\left[\left(Dr^2+s^2\right)-\sqrt{D}\cdot 2rs\right]}{\left(Dr^2-s^2\right)^2}$

Choice (iii):

$$1=\frac{\left[\left(2k^2+2k+\frac{(D+1)}{2}\right)+\sqrt{D}(\cdot 2k+1\right]\left[\left(2k^2+2k+\frac{(D+1)}{2}\right)-\sqrt{D}\cdot(2k+1)\right]}{\left(2k^2+2k-\frac{(D-1)}{2}\right)^2}$$

Choice(iv):

$$1=\frac{\left[(2k^2+2k)D+\frac{(D+1)}{2}+\sqrt{D}(\cdot 2k+1\right]\left[(2k^2+2k)D+\frac{(D+1)}{2}-\sqrt{D}\cdot(2k+1)\right]}{\left|(2k^2+2k)D+\frac{(D-1)}{2}\right|^2}$$

It is worth mentioning that the repetition of the process as in method 4 for each of the above choices leads to different set of solutions to (1).

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let $\left(x_0, y_{0,} z_0\right)$ be any given solution to (1)

Formula: 1

Let $\left(x_1, y_{1,} z_1\right)$ given by

$$x_1=-(D-1)x_0+h,\ y_1=(D-1)y_0,\ z_1=(D-1)z_0+h, \tag{11}$$

be the 2nd solution to (1).Using (11) in (1) and simplifying, one obtains

$$h=2Dx_0+2z_0$$

In view of (11), the values of x_1 and z_1 is written in the matrix form as

$$(x_1, z_1)^t = M(x_0, z_0)^t$$

where

$$M = \begin{pmatrix} D+1 & 2 \\ 2D & D+1 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = D+1+2\sqrt{D}, \quad \beta = D+1-2\sqrt{D}$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha-\beta)}(M-\beta I) + \frac{\beta^n}{(\beta-\alpha)}(M-\alpha I), I = 2\times 2 \text{ identity matrix}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = \left(\frac{\alpha^n+\beta^n}{2}\right)x_0 + \left(\frac{\alpha^n-\beta^n}{2\sqrt{D}}\right)z_0 ,$$

$$y_n = (D-1)^n y_0 ,$$

$$z_n = \frac{\sqrt{D}}{2}(\alpha^n - \beta^n)x_0 + \left(\frac{\alpha^n+\beta^n}{2}\right)z_0$$

Formula: 2

Let $(x_1, y_{1,} z_1)$ given by

$$x_1 = 3x_0,\ y_1 = 3y_0 + h,\ z_1 = 2h - 3z_0, \tag{12}$$

be the 2^{nd} solution to (1).Using (12) in (1) and simplifying, one obtains

$$h = 2y_0 + 4z_0$$

In view of (12), the values of y_1 and z_1 is written in the matrix form as

$$(y_1, z_1)^t = M^n (y_0, z_0)^t$$

where

$$M = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions y_n, z_n given by

$$(y_n, z_n)^t = M^n (y_0, z_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = 1,\ \beta = 9$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 3^n x_0\ ,$$

$$y_n = \left(\frac{9^n + 1}{2}\right) y_0 + \left(\frac{9^n - 1}{2}\right) z_0\ ,$$

$$z_n = \frac{(9^n - 1)}{2} y_0 + \left(\frac{9^n + 1}{2}\right) z_0$$

Formula: 3

Let $(x_1, y_{1,} z_1)$ given by

$$x_1 = -(D+1)x_0 + h,\ y_1 = -(D+1)y_0 + h,\ z_1 = (D+1)z_0, \qquad (13)$$

be the 2^{nd} solution to (1).Using (13) in (1) and simplifying, one obtains

$$h = 2Dx_0 + 2y_0$$

In view of (13), the values of x_1 and y_1 is written in the matrix form as

$$(x_1, y_1)^t = M^n (x_0, y_0)^t$$

where

$$M = \begin{pmatrix} D-1 & 2 \\ 2D & -D+1 \end{pmatrix} \text{and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n (x_0, y_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = D+1 \; \beta = -(D+1)$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = \frac{\alpha^n D + \beta^n}{D+1} x_0 + \frac{\alpha^n - \beta^n}{D+1} y_0,$$
$$y_n = \frac{D(\alpha^n - \beta^n)}{D+1} x_0 + \frac{\alpha^n + \beta^n D}{D+1} y_0,$$
$$z_n = (D+1)^n z_0$$

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