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A SEARCH ON INTEGER SOLUTIONS TO NON-HOMOGENEOUS TERNARY CUBIC EQUATION $9(x^2-y^2)+x+y=4z^3$

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ABSTRACT

This paper concerns with the problem of obtaining non-zero distinct integer solutions to non-homogeneous ternary cubic Diophantine equation $9(x^2-y^2)+x+y=4z^3$ ***. A few relations between the solutions are presented.***

KEY WORDS: ***ternary cubic, non-homogeneous cubic, integer solutions***

NOTATION

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

INTRODUCTION

The cubic Diophantine equations are rich in variety and offer an unlimited field for research [1,2]. In particular, refer [3-16] for a few problems on cubic equation with 3 unknowns. This paper concerns with an interesting non-homogeneous cubic Diophantine equation with three unknowns given by $9(x^2-y^2)+x+y=4z^3$ for determining its infinitely many non-zero distinct integral solutions. A few relations between the solutions are presented.

METHOD OF ANALYSIS

Consider the non-homogeneous cubic Diophantine equation

$$9(x^2-y^2)+x+y=4z^3 \tag{1}$$

Different ways of solving (1) are presented below:

WAY: 1

The substitution of the linear transformations

$$x=ku+v,\ y=ku-v,\ z=u \tag{2}$$

in (1) leads to

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$$u^2 = 18v + 1 \qquad (3)$$

whose smallest positive integer solution is

$$u_0 = 19 , v_0 = 20$$

Assume that

$$u_1 = h - u_0 , v_1 = v_0 + h \qquad (4)$$

be the second solution to(3).Substituting (4) in (3) and simplifying
, note that

$$h = 2u_0 + 18$$

In view of (4), one obtains

$$u_1 = u_0 + 18 , v_1 = 2u_0 + v_0 + 18$$

Repeating the above process again and again, the general solution (u_n , v_n) to (3) is given by

$$u_n = u_0 + 18n , v_n = 2nu_0 + v_0 + 18n^2$$

In view of (2), the corresponding integer solutions to (1) are given by

$$x_n = (k + 2n)u_0 + v_0 + 18n(k + n),$$
$$y_n = (k - 2n)u_0 - v_0 + 18n(k - n),$$
$$z_n = u_0 + 18n$$

Properties:

(i) $x_n - y_n - 76n - 40$ **is** a perfect square

(ii) $x_n - y_n - 72t_{3,n} \cong 0 (\mathrm{mod}\, 40)$

(iii) $x_n - y_n - 72t_{74,n} \cong 71 (\mathrm{mod}\, 111)$

(iii) $x_n - y_n - 72t_{3,n} - 2z_n + 78 \cong 0 (\mathrm{mod}\, 4)$

WAY: 2

Considering

$$x = ku + v^2 , y = ku - v^2 , z = u \qquad (5)$$

in (1) ,it reduces to the Pellian equation

$$u^2 = 18v^2 + 1$$

whose general solution $(\boldsymbol{u_n} , \boldsymbol{v_n})$ is given by

$$u_n = \frac{f_n}{2} , v_n = \frac{g_n}{2\sqrt{18}}, \ n = 0,1,2,...$$

where

$$f_n = (17 + 4\sqrt{18})^{n+1} + (17 - 4\sqrt{18})^{n+1} ,$$
$$g_n = (17 + 4\sqrt{18})^{n+1} - (17 - 4\sqrt{18})^{n+1} ,$$

In view of (5), the corresponding integer solutions to (1) are given by

$$x_n = \frac{k}{2} f_n + \frac{1}{12} g_n^2$$
$$y_n = \frac{k}{2} f_n - \frac{1}{72} g_n^2$$
$$z_n = \frac{1}{2} f_n$$

Properties:

1. $z_n{}^2 - 9(x_n - y_n) = 1$
2. $(x_{2n+1} + y_{2n+1}) - 36k(x_n - y_n) = 2k$
3. $(x_n + y_n)^2 - 36(x_n - y_n)k^2 = 4k^2$
4. Each of the following expressions is a perfect square:

- $36(x_n - y_n) + 4$, $k(x_{2n+1} + y_{2n+1} + 2k)$
- $k^2(x_{3n+2} + y_{3n+2} + 3(x_n + y_n))$ is a cubical integer
- $k^3(x_{4n+3} + y_{4n+3} + 4(x_{2n+1} + y_{2n+1} + 2k) - 2k)$ is a bi-quadratic integer

Way :3

Introduction of the transformations

$$x = ku^2 + v^2 , y = ku^2 - v^2 , z = u$$

in (1) leads to

$$u = 18v^2 + 1$$

Thus, the corresponding integer solutions to (1) are obtained as

$$x = k(18k^2 + 1)^2 + k^2 ,$$
$$y = k(18k^2 + 1)^2 - k^2 ,$$
$$z = 18k^2 + 1$$

Properties:

(i) $z = 9(x - y) + 1$

(ii) Each of the following expressions is a nasty number:

$3(x - y)$,$3(z - 1)$,$3k(x + y + 4kz + 2k)$

CONCLUSION

In this paper, we have presented different sets of non-zero distinct integer solutions to the ternary cubic equation $9(x^2 - y^2) + x + y = 4z^3$. As the cubic Diophantine equations are rich in variety, one may search for the other choices of equations along with their solutions and relations among the solutions.

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REFERENCES

1. *L.E. Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing company, NewYork, 1952.*
2. *L.J. Mordell, Diophantine equations, Academic press, New York, 1969.*
3. *M.A. Gopalan, G. Sangeetha, "On the ternary cubic Diophantine equation* $y^2 = Dx^2 + z^3$ *", Archimedes J.Math 1(1), 2011, 7-14.*
4. *M.A. Gopalan, B. Sivakami, "Integral solutions of the ternary cubic equation* $4x^2 - 4xy + 6y^2 = \left((k+1)^2 + 5\right)w^3$ *",Impact J.Sci.Tech, Vol.6, No.1, 2012, 15-22.*
5. *M.A. Gopalan, B. Sivakami, "On the ternary cubic Diophantine equation* $2xz = y^2(x+z)$ *", Bessel J.Math 2(3), 2012, 171-177.*
6. *S. Vidyalakshmi, T.R. Usharani, M.A. Gopalan, "Integral solutions of non-homogeneous ternary cubic equation* $ax^2 + by^2 = (a+b)z^3$ *", Diophantus J.Math 2(1), 2013, 31-38.*
7. *M.A. Gopalan, K. Geetha, "On the ternary cubic Diophantine equation* $x^2 + y^2 - xy = z^3$ *", Bessel J.Math., 3(2), 2013,119-123.*
8. *M.A. Gopalan, S. Vidhyalakshmi, A.Kavitha "Observations on the ternary cubic equation* $x^2 + y^2 + xy = 12z^3$ *", Antartica J.Math 10(5), 2013, 453-460.*
9. *M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi, "Lattice points on the non-homogeneous cubic equation* $x^3 + y^3 + z^3 + (x+y+z) = 0$ *", Impact J.Sci.Tech, Vol.7, No.1, 2013, 21-25.*
10. *M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi "Lattice points on the non-homogeneous cubic equation* $x^3 + y^3 + z^3 - (x+y+z) = 0$ *", Impact J.Sci.Tech, Vol.7, No1, 2013, 51-55,*
11. *M.A. Gopalan, S. Vidhyalakshmi, S. Mallika, "On the ternary non-homogenous cubic equation* $x^3 + y^3 - 3(x+y) = 2\left(3k^2 - 2\right)z^3$ *", Impact J.Sci.Tech, Vol.7, No.1, 2013, 41-45.*
12. *S. Vidhyalakshmi, M.A. Gopalan, S. Aarthy Thangam, "On the ternary cubic Diophantine equation* $4(x^2 + x) + 5(y^2 + 2y) = -6 + 14z^3$ *" International Journal of Innovative Research and Review (JIRR), Vol 2(3)., pp 34-39, July-Sep 2014*
13. *M.A. Gopalan, N. Thiruniraiselvi and V. Kiruthika, "On the ternary cubic diophantine equation* $7x^2 - 4y^2 = 3z^3$ *", IJRSR, Vol.6, Issue-9, Sep-2015, 6197-6199.*
14. *M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, J. Maheswari, "On ternary cubic diophantine equation* $3\left(x^2 + y^2\right) - 5xy + x + y + 1 = 12z^3$ *", International Journal of Applied Research, 1(8), 2015, 209-212.*
15. *R. Anbuselvi, K. Kannaki, "On ternary cubic diophantine equation* $3\left(x^2 + y^2\right) - 5xy + x + y + 1 = 15z^3$ *", IJSR, Vol.5, Issue-9, Sep 2016, 369-375.*
16. *G. Janaki, C. Saranya, "Integral solutions of the ternary cubic equation* $3\left(x^2 + y^2\right) - 4xy + 2(x+y+1) = 972z^3$ *", IRJET, Vol.04, Issue 3, March 2017, 665-669.*