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NEW GENERALIZED CIESARO SPACE WITH SOME TOPOLOGICAL PROPERTIES

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ABSTRACT

The sequence space introduced by M. Et and have studied its various properties. The aim of the present paper is to introduce the new pranormed generalized difference sequence space. $[f, g, p, u](\Delta_n^r), [f, g, p, u]_0(\Delta_n^r)$ ***and*** $[f, g, p, u]_\infty(\Delta_n^r)$***, We give some topological properties and inclusion relations on these spaces.***

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1. INTRODUCTION

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers respectively. Let ω denote the space of all sequences (real or complex). Let l_∞ and c be Banach spaces of bounded and convergent sequences $x = \{x_n\}_{n=0}^{\infty}$ with supremum norm $\|x\| = \sup_n |x_n|$. Let T denote the shift operator on ω, that is, $Tx = \{x_n\}_{n=1}^{\infty}$, $T^2 x = \{x_n\}_{n=2}^{\infty}$ and so on. A Banach limit L is defined on l_∞ as a non-negative linear functional such that L is invariant *i.e.*, $L(Sx) = L(x)$ and $L(e) = 1, e = (1,1,1,...)$ (see, [12]).

Lorentz (see, [12]), called a sequence $\{x_n\}$ almost convergent if all Banach limits of x, $L(x)$, are same and this unique Banach limit is called F-limit of x. In his paper, Lorentz proved the following criterian for almost convergent sequences.

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A sequence $x=\{x_n\}\in l_\infty$ is almost convergent with F -limit $L(x)$ if and only if

$$\lim_{m\to\infty} t_{mn}(x)=L(x)$$

where, $t_{mn}(x)=\frac{1}{m}\sum_{j=0}^{m-1}T^j x_n, (T^0=0)$ uniformly in $n\geq 0$.

We denote the set of almost convergent sequences by f.

Several authors including Duran (see, [5]), Ganie et al (see, [1, 2, 3, 4, 22]), King (see, [10]), Lorentz (see, [12]) and many others have studied almost convergent sequences. Maddox (see, [15, 14]) has defined x to be strongly almost convergent to a number α if

$$\lim_{n}\frac{1}{n}\sum_{k=1}^{n}|x_{k+m}-\alpha|=0, \text{ uniformly in m.}$$

By $[f]$ we denote the space of all strongly almost convergent sequences. It is easy to see that $c\subset f\subset [f]\subset ł_\infty$.

The concept of paranorm is related to linear matric spaces. It is a generalization of that of absolute value. Let X be a linear space. A function $P: x\to R$ is called a paranorm, if (see, [13, 24])

$(p.1)$ $p(0)\geq 0$

$(p.2)$ $p(x)\geq 0 \;\forall\; x\in X$

$(p.3)$ $p(-x)=p(x)\;\forall\; x\in X$

$(p.4)$ $p(x+y)\leq p(x)+p(y)\;\forall\; x,y\in X$ (triangle inequality)

$(p.5)$ if (λ_n) is a sequence of scalars with $\lambda_n\to\lambda$ $(n\to\infty)$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0 \quad (n \to \infty)$, then $p(x_n\lambda_n - x\lambda) \to 0 \quad (n \to \infty)$,(continuity of multiplication of vectors).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total. It is well known that the metric of any linear metric space is given by some total paranorm (see, [15]).

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \le \sup_k p_k = H < \infty$ and let $D = \max(1, 2^{H-1})$. For $a_k, b_k \in \mathbb{C}$. We have (see, [13, 14]) that

$$| a_k + b_k |^{p_k} \le D\{| a_k |^{p_k} + | b_k |^{p_k} .\} \qquad (1)$$

Nanda (see, [18, 19]) defined the following:

$$[f, p] = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - \alpha|^{p_k} = 0 \text{ uniformly in m} \right\},$$

$$[f, p]_0 = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m}|^{p_k} = 0 \text{ uniformly in m} \right\},$$

$$[f, p]_\infty = \left\{ x : \sup_{m,n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+m}|^{p_k} < \infty \right\}.$$

The difference sequence spaces,

$$X(\Delta) = \{ x = (x_k) : \Delta x \in X \},$$

where $X = l_\infty, c$ and c_0, were studied by Kizmaz (see, [11]).

It was further generalized by Et and Çolak (see, [8]), Ganie et al (see, [3]), Sengönül (see, [21]) and many others.

Further, it was Tripathy et al (see, [23]) generalized the above notions and unified these as follows:

$$\Delta_n^m x_k = \left\{x \in \omega : (\Delta_n^m x_k) \in Z\right\},$$

where

$$\Delta_n^m x_k = \sum_{\mu=0}^{n} (-1)^{\mu} \binom{n}{r} x_{k+m\mu},$$

and

$$\Delta_n^0 x_k = x_k \forall\, k \in \mathrm{N}.$$

Recently, M. Et (see, [6]) defined the following:

$$[f, p](\Delta^r) = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \left[f\left(\left| \Delta^r x_{k+m} - \alpha \right| \right) \right]^{p_k} = 0, \text{uniformly in m} \right\},$$

$$[f, p]_0(\Delta^r) = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \left[f\left(\left| \Delta^r x_{k+m} \right| \right) \right]^{p_k} = 0, \text{uniformly in m} \right\},$$

$$[f, p]_\infty(\Delta^r) = \left\{ x = (x_k) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[f\left(\left| \Delta^r x_{k+m} \right| \right) \right]^{p_k} < \infty, \text{uniformly in m} \right\}.$$

Following Maddox (see, [16])and Ruckle (see, [20]), a modulus function g is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $g(x) = 0$ if and only if $x = 0$,

(ii) $g(x + y) \leq g(x) + g(y)\ \forall x, y \geq 0$

(iii) g is increasing,

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(iv) g if continuous from right at $x = 0$.

Maddox (see, [15])introduced and studied the following sets:

$$f_0 = \{x \in \omega : \lim_n \frac{1}{n}\sum_{k=1}^{n} | x_{k+m} | = 0 \, uniformly \, in \, m\}$$

$$f = \{x \in \omega : x - le \in f_0 \; for \, some \, in \, l \in \mathbb{C}\}$$

of sequences that are strongly almost convergent to zero and strongly almost convergent.

Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \le \sup_k p_k = M$ and $H = \max(1, M)$.

2. MAIN RESULTS

In the present paper, we define the spaces $[f, g, p, u](\Delta_n^r), [f, g, p, u]_0(\Delta_n^r)$ and $[f, g, p, u]_\infty(\Delta_n^r)$, where $u = (u_k)$ is such that $u_k \neq 0 \; \forall k$, as follows:

$$[f, g, p, u](\Delta_n^r) = \left\{ x = (x_k) : \lim_n \frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k \Delta_n^r x_{k+m} - \alpha\right|\right)\right]^{p_k} = 0, \text{uniformly in m} \right\},$$

$$[f, g, p, u]_0(\Delta_n^r) = \left\{ x = (x_k) : \lim_n \frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k \Delta_n^r x_{k+m}\right|\right)\right]^{p_k} = 0, \text{uniformly in m} \right\},$$

$$[f, g, p, u]_\infty(\Delta_n^r) = \left\{ x : \sup_n \frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k \Delta_n^r x_{k+m}\right|\right)\right]^{p_k} < \infty, \text{uniformly in m} \right\},$$

where (p_k) is any bounded sequence of positive real numbers.

Theorem 1: Let (p_k) be any bounded sequence and g be any modulus function. Then

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$[f,g,p,u](\Delta_n^r),[f,g,p,u]_0(\Delta_n^r)$ and $[f,g,p,u]_\infty(\Delta_n^r)$ are linear space over the set of complex numbers.

Proof: We shall prove the result for $[f,g,p,u]_0(\Delta_n^r)$ and the others follows on similar lines. Let $x,y\in[f,g,p,u]_0(\Delta_n^r)$. Now for $\alpha,\beta\in C$, we can find positive numbers A_α,B_β such that $|\alpha|\le A_\alpha$ and $|\beta|\le B_\beta$. Since f is sub-additive and Δ_n^r is linear

$$\frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r\left(\alpha x_{k+m}+\beta y_{k+m}\right)\right|\right)\right]^{p_k}$$

$$\le\frac{1}{n}\sum_{k=1}^{n}\left[g\left(|\alpha|\left|u_k\Delta_n^r x_{k+m}\right|\right)+g\left(|\beta|\left|u_k\Delta_n^r\beta y_{k+m}\right|\right)\right]^{p_k}$$

$$\le D(A_\alpha)^H\frac{1}{n}\sum_{k=1}^{n}\left[g\left(|\alpha|\left|u_k\Delta_n^r x_{k+m}\right|\right)\right]^{p_k}$$

$$+D(B_\beta)^H\frac{1}{n}\sum_{k=1}^{n}\left[g\left(|\alpha|\left|u_k\Delta_n^r x_{k+m}\right|\right)\right]^{p_k}\to 0$$

as $n\to\infty$, uniformly in m. This proves that $[f,g,p,u_k]_0(\Delta_n^r)$ is linear and the result follows.
W

Theorem 2: Let g be any modulus function. Then

$[f,g,p,u](\Delta_n^r)\subset[f,g,p,u]_\infty(\Delta_n^r)$ and $[f,g,p,u]_0(\Delta_n^r)\subset[f,g,p,u]_\infty(\Delta_n^r)$.

Proof: We shall prove the result for $[f,g,p,u](\Delta_n^r)\subset[f,g,p,u]_\infty(\Delta_n^r)$ and the second shall be proved on similar lines. Let $x\in[f,g,p,u](\Delta_n^r)$. Now, by definition of g, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m}\right|\right)\right]^{p_k}=\frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m}-L+L\right|\right)\right]^{p_k}$$

$$\leq \frac{D}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m} - L\right|\right)\right]^{p_k} + \frac{D}{n}\sum_{k=1}^{n}\left[g\left(\left|L\right|\right)\right]^{p_k}.$$

Thus, for any number L, there exists a positive integer K_L such that $|L| \leq K_L$, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m}\right|\right)\right]^{p_k} = \frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m} - L + L\right|\right)\right]^{p_k}$$

$$\leq \frac{D}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m} - L\right|\right)\right]^{p_k} + \frac{D}{n}\left[K_L g(1)\right]^{p_k}\sum_{k=1}^{n}1.$$

Since, $x \in [f, g, p, u](\Delta_n^r)$, we have $x \in [f, g, p, u]_\infty(\Delta_n^r)$ and the proof of second result follows.
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Theorem 3: $[f,, g, p]_0(\Delta_n^r)$ is a paranormed space with

$$h_\Delta(x) = \sup_{m,n}\left(\frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m}\right|\right)\right]^{p_k}\right)^{\frac{1}{H}}.$$

Proof: From Theorem 2, for each $x \in [f, g, p, u]_0(\Delta_n^r)$, $h_\Delta(x)$ exists. Also, it is trivial that $h_\Delta(x) = h_\Delta(-x)$ and $\Delta_n^r x_{k+m} = 0$ for $x = 0$. Since, $h(0) = 0$, we have $h_\Delta(x) = 0$ for $x = 0$. Since, $\frac{p_k}{M} \leq 1$ for $M \geq 1$, therefore, by Minkowski's inequality and by definition of g for each n that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m} + \Delta_n^r y_{k+m}\right|\right)\right]^{p_k}\right)^{\frac{1}{H}}$$

$$\leq \left(\frac{1}{n}\sum_{k=1}^{n}\left[g\left(\left|u_k\Delta_n^r x_{k+m}\right|\right) + g\left(\left|u_k\Delta_n^r y_{k+m}\right|\right)\right]^{p_k}\right)^{\frac{1}{H}}$$

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$$\leq \left(\frac{1}{n} \sum_{k=1}^{n} \left[g\left(\left| u_k \Delta_n^r x_{k+m} \right| \right) \right]^{p_k} \right)^{\frac{1}{H}} + \left(\frac{1}{n} \sum_{k=1}^{n} \left[g\left(\left| u_k \Delta_n^r y_{k+m} \right| \right) \right]^{p_k} \right)^{\frac{1}{H}},$$

which shows that $h_\Delta(x)$ is sub-additive. Further, let α be any complex number. Therefore, we have by definition of g, we have

$$h_\Delta(\alpha x) = \sup_{m,n} \left(\frac{1}{n} \sum_{k=1}^{n} \left[g\left(\left| u_k \Delta_n^r \alpha x_{k+m} \right| \right) \right]^{p_k} \right)^{\frac{1}{H}} \leq S_\alpha^{\frac{H}{M}} h_\Delta(x),$$

where, S_α is an integer such that $\alpha < S_\alpha$. Now, let $\alpha \to 0$ for any fixed x with $h_\Delta(x) \neq 0$. By definition of g for $|\alpha| < 1$, we have for $n > N(\varepsilon)$ that

$$\frac{1}{n} \sum_{k=1}^{n} \left[g\left(\left| u_k \Delta_n^r x_{k+m} \right| \right) \right]^{p_k} < \varepsilon. \tag{2}$$

As g is continuous, we have, for $1 \leq n \leq N$ and by choosing α so small that

$$\frac{1}{n} \sum_{k=1}^{n} \left[g\left(\left| u_k \Delta_n^r x_{k+m} \right| \right) \right]^{p_k} < \varepsilon. \tag{3}$$

Consequently, (2) and (3) gives that $h_\Delta(\alpha x) \to 0$ as $\alpha \to 0$. W

Theorem 4: The spaces $[f, g, p, u](\Delta_n^r), [f,, g, p]_0(\Delta_n^r)$ and $[f,, g, p]_\infty(\Delta_n^r)$ are not solid in general.

Proof : To show that the spaces $[f, g, p, u](\Delta_n^r), [f,, g, p]_0(\Delta_n^r)$ and $[f,, g, p]_\infty(\Delta_n^r)$ are not solid in general, we consider the following example.

Let $p_k = 1 = u_k$ for all k and $g(x) = x$ with $r = 1 = n$. Then, $(x_k) = (k) \in [f,, g, p]_\infty(\Delta_n^r)$ but $(\alpha_k x_k) \notin [f,, g, p]_\infty(\Delta_n^r)$ when $\alpha_k = (-1)^k$ for all $k \in \mathrm{N}$. Hence is result follows. W

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From above Theorem, we have the following corollary.

Corollary 5: The spaces $[f,g,p,u](\Delta_n^r), [f,g,p]_0(\Delta_n^r)$ and $[f,,g,p]_\infty(\Delta_n^r)$ are not perfect.

Theorem 6: The spaces $[f,g,p,u](\Delta_n^r), [f,g,p,u]_0(\Delta_n^r)$ and $[f,g,p,u]_\infty(\Delta_n^r)$ are not symmetric in general.

Proof : To show that the spaces $[f,g,p,u](\Delta_n^r), [f,g,p,u]_0(\Delta_n^r)$ and $[f,g,p,u]_\infty(\Delta_n^r)$ are not perfect in general, To show this, let us consider $p_k = 1 = u_k$ for all k and $g(x) = x$ with $n = 1$. Then, $(x_k) = (k) \in [f,g,p,u]_\infty(\Delta_n^r)$ Let the re-arrangement of (x_k) be (y_k) where (y_k) is defined as follows,

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_1 6, x_6, x_2 5, x_7, x_3 6, x_8, x_4 9, x_1 0, \ldots\}.$$

Then, $(y_k) \notin [f,g,p,u]_\infty(\Delta_n^r)$ and this proves the result.

REFERENCES

1. *A. H. Ganie, N. A. Sheikh, Infinite matrices and almost convergence, Filomat, 29(6)(2015), 1183-1188..*
2. *A. H. Ganie and N. A. Sheikh, Infinite matrices and almost bouned sequences, Vietnam Journal of Mathematics, 42(2)(2014), 153-157.*
3. *A. H. Ganie and N. A. Sheikh, On some new sequence space of non-absolute type and matrix transformations, Jour. Egyptain Math. Soc., 21(2013), 34-40.*
4. *A. H. Ganie, N. A. Sheikh and T. Jalal, On some new type of invariant means with respect to modulus function, Int. J. Mod. Math. Sc, 13(1)(2015), 210-216.*
5. *J. P. Duran, Infinite matrices and almost convergence, math. Zeit., 128(1972), 75-83.*
6. *M. Et, Strongly almost summable difference sequences of order* m *defined by modulus, Studia Scientiarum Math. Hung., 40(2003), 463-476.*
7. *M. Et and M. Basarir, On some new generalized difference sequence spaces, Periodica Mathematica Hungarica 35(3)(1997), 169-175.*
8. *M. Et and R. Çolak, On some generalised difference sequence spaces, Soochow J. Math. 21(1995), 377-386.*
9. *P. Kampthan and M. Gupta, Sequence spaces and series, Marcel Dekkar Inc., (1981).*
10. *J. P. King, Almost summable sequences, Proc. Ammer. Math. Soc., 16(1966), 1219-1225.*
11. *H. Kizmaz, On certain sequence spaces, Canad. Math. Bull. 24(1981), 169-176.*
12. *G. G. Lorentz, A contribution to the theory of divergent sequences, Acta Math., 80(1948), 167-190.*

13. I. J. Maddox, Elements of Functionls Analysis, Cambridge Univ. Press, (1970).
14. I. J. Maddox, On strong almost convergence, Math. Proc. Camb. Phil. Soc., 85(1979), 345-350.
15. I. J. Maddox, Spaces of strongly summable sequences, Qurt. J. Math., 18(1967), 345-355.
16. I. J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Camb. Phil. Soc., 100(1986), 161-166.
17. M. Mursaleen, Generalized spaces of difference sequences, Jour. Math. Anal. App., 203(1996), 738-745.
18. S. Nanda, Strongly almost convergent sequences, Bull. Cal. Math. Soc., 76(1984), 236-240.
19. S. Nanda, Strongly summable and strongly almost convergent sequences, Acta Math. Hung., 49(1987), 71-76.
20. W. H. Ruckle, FK spaces in which the sequence coordinate verctors in bounded, Canad. J. Math., 25(1973), 973-978.
21. M. Sengönül and F. Ba S ar, Some new Cesàro sequences spaces of non-absolute type, which include the spaces c_o and c, Soochow J. Math., 1(2005), 107-119.
22. N. A. Sheikh and A. H. Ganie,On the space of λ-convergent sequence and almost convergence, Thai Journal of Math., 2(11) (2013), 393-398.
23. Tripathy, B. C., Esi, A., and Tripathy, B., On a new type of generalized difference Cesaro sequence spaces, 31(3)(2005), 33-340.
24. A. Wilansky, Summability through Functional Analysis, North Holland Mathematics Studies, Oxford, (1984).