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ON SOME SPECIAL FEATURES OF THE HENON MAPPING

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ABSTRACT

In this paper we have studied Henon map as a two dimensional discrete non-linear dynamical system. We have studied some special features of the Henon mapping. We have obtained the fixed points as we as the periodic orbits of the Henon map and obtained some results regarding their stability. Also, we have obtained the graphs of the iterates for different initial conditions and shown the presence of the chaotic attractor for different parameter values.

KEYWORDS: ***Dynamical system, Henon map, fixed points, periodic orbits, strange attractor, chaos***

1. INTRODUCTION

An unpredictable and very strange dynamics is observed in very complex dynamical systems [3, 4, 6] appearing in nature can also be observed in very simple nonlinear dynamical systems. The simplest of such dynamical systems is the tent function [2, 5] $T_\mu(x)$ defined on the unit interval $[0, 1]$ by

$$T_\mu(x) = \begin{cases} \mu x & 0 \le x \le \frac{1}{2} \\ \mu(1-x) & \frac{1}{2} \le x \le 1 \end{cases},$$

where μ is a parameter with $0 \le \mu \le 2$. Many authors have proved that the tent mapping has a period three cycle [7, 9] given by $\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$ for $\mu = 2$. The existence of period 3-cycle is one of the indications of the chaotic phenomenon [10, 12]

In this paper we will point out some of the important features of the Henon map. The Henon map was discovered by the French astronomer Michel Henon in 1975 who showed that a strange attractor can also be found in a two dimensional quadratic mapping. The Henon map $f: R^2 \to R^2$ is defined by $f(x,\ y) = (1 - \alpha x^2 + y,\ \beta x)$, where α and β are parameters. If we set $y = 0$, $\beta = 1$ and $x = t$, then $f(x,\ y) = f(t,\ 0) = (1 - \alpha t^2,\ t)$. Thus the image of the real line under the Henon map is the parabola whose parametric equations are $x = 1 - \alpha x^2, y = t$.

2. FEATUES OF THE HENON MAP

In this section we will study the important features of the Henon map. The next theorem gives us the eigenvalues of the differential [15] of $f(x,\ y)$.

2.1 Theorem: *If $2\alpha x^2 + \beta \ge 0$, then the eigenvalues of the differential $D[f(x,\ y)]$ of $f(x,\ y)$ are real and given by $-\alpha x \pm \sqrt{\alpha^2 x^2 + \beta}$. The mapping $f(x,\ y)$ is area contracting [10, 11] if $0 \le \beta < 1$*

Proof: The Henon map $f: R^2 \to R^2$ is defined by $f(x,\ y) = (1 - \alpha x^2 + y,\ \beta x)$, where α and β are parameters. We can write

$f(x,\ y) = (1 - \alpha x^2 + y,\ \beta x) = (f_1(x,\ y),\ f_2(x,\ y)),$

where $f_1(x,\ y) = 1 - \alpha x^2 + y$ and $f_2(x,\ y) = \beta x$.

Hence the differential of f at the point $(x,\ y)$ is given by the matrix

$$D[f(x,\ y)] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} -2\alpha x & 1 \\ \beta & 0 \end{bmatrix}$$

The eigenvalues of the differential $D[f(x,\ y)]$ *of* $f(x,\ y)$ are given by

$$det.(\ D[f(x,\ y)] - \lambda I) = 0$$

Hence

$$\begin{vmatrix} -2\alpha x - \lambda & 1 \\ \beta & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 2\alpha x\lambda - \beta = 0$$

$$\Rightarrow \lambda = \frac{-2\alpha x \pm \sqrt{4\alpha^2 x^2 + 4\beta}}{2} = -\alpha x \pm \sqrt{\alpha^2 x^2 + \beta}$$

Thus the eigenvalues are real if $\alpha^2 x^2 + \beta \geq 0$.

The Jacobian of $f(x,\ y)$ is given by

$J = det.(\ D[f(x,\ y)]) = \begin{vmatrix} -2\alpha x & 1 \\ \beta & 0 \end{vmatrix} = 0 - \beta = -\beta$, which is a constant.

Hence the mapping $f(x,\ y)$ is area contracting if $0 \leq \beta < 1$.

This completes the proof.

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2.2 Fixed points of the Henon Map
Now we will obtain the fixed points [12] of the Henon map and study their stability [13]. The fixed points of $f(x,y)$ are obtained by solving the equation $f(x,y) = (x,y)$, which gives

$$1 - \alpha x^2 + y = x, \qquad \beta x = y$$

$$\Rightarrow 1 - \alpha x^2 + \beta x = x$$

$$\Rightarrow 1 - \alpha x^2 + (\beta - 1)x = 0$$

$$\Rightarrow \alpha x^2 + (1 - \beta)x - 1 = 0$$

$$\Rightarrow x = \frac{-(1-\beta) \pm \sqrt{(1-\beta)^2 + 4\alpha}}{2\alpha}$$

$$\Rightarrow x = \frac{1}{2\alpha}\left[\beta - 1 \pm \sqrt{(1-\beta)^2 + 4\alpha}\right], \quad \alpha \neq 0$$

Such an x exists if $(1-\beta)^2 + 4\alpha \geq 0$ $i.e.$ $\alpha \geq \frac{-1}{4}(1-\beta)^2$.

Thus the fixed points are given by $P \equiv \left(\frac{1}{2\alpha}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 4\alpha}\right], \ \frac{\beta}{2\alpha}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 4\alpha}\right]\right)$ and

$$Q \equiv \left(\frac{1}{2\alpha}\left[\beta - 1 - \sqrt{(1-\beta)^2 + 4\alpha}\right], \ \frac{\beta}{2\alpha}\left[\beta - 1 - \sqrt{(1-\beta)^2 + 4\alpha}\right]\right)$$

where $\alpha \geq \frac{-1}{4}(1-\beta)^2$.

2.3 Theorem: *If* $\alpha \in \left(\frac{-(1-\beta)^2}{4}, \ \frac{3(1-\beta)^2}{4}\right) = I$, *then the fixed point* $P \equiv \left(\frac{1}{2\alpha}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 4\alpha}\right], \ \frac{\beta}{2\alpha}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 4\alpha}\right]\right)$ *is an attracting fixed point and* $Q \equiv \left(\frac{1}{2\alpha}\left[\beta - 1 - \sqrt{(1-\beta)^2 + 4\alpha}\right], \ \frac{\beta}{2\alpha}\left[\beta - 1 - \sqrt{(1-\beta)^2 + 4\alpha}\right]\right)$ *is a saddle point [14] of the Henon map* $f(x, \ y)$.

Proof: Let $P \equiv (p_1, \ p_2)$, where $p_1 = \frac{1}{2\alpha}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 4\alpha}\right]$ and

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$$p_2 = \frac{\beta}{2\alpha}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 4\alpha}\right]$$

As $\sqrt{(1-\beta)^2 + 4\alpha} > 0$, it follows that $2\alpha p_1 > \beta - 1$ *i.e.* $2\alpha p_1 + 1 > \beta$.

Now assume that $\alpha \in \left(\frac{-(1-\beta)^2}{4}, \ \frac{3(1-\beta)^2}{4}\right) = I$ so that $\frac{-(1-\beta)^2}{4} < \alpha$

$\Rightarrow 4\alpha + (1-\beta)^2 > 0$. This proves that p_1 is not an imaginary number. Hence ${p_1}^2 \geq 0$

Also, we have

$(\alpha p_1 + 1)^2 = \alpha^2 {p_1}^2 + 2\alpha p_1 + 1 > \alpha^2 {p_1}^2 + \beta > 0$

$\Rightarrow \alpha p_1 + 1 > \sqrt{\alpha^2 {p_1}^2 + \beta}$

$\Rightarrow -\alpha p_1 + \sqrt{\alpha^2 {p_1}^2 + \beta} < 1$

Thus we have proved that $0 \leq -\alpha p_1 + \sqrt{\alpha^2 {p_1}^2 + \beta} < 1$

Similarly by assuming that $\alpha < \frac{3(1-\beta)^2}{4}$, it can be proved that $-1 < -\alpha p_1 - \sqrt{\alpha^2 {p_1}^2 + \beta} < 0$

Combining these two statements, we have $\left|-\alpha p_1 \pm \sqrt{\alpha^2 {p_1}^2 + \beta}\right| < 1$

We know that the eigenvalues of the matrix

$$D[f(x,\ y)] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -2\alpha x & 1 \\ \beta & 0 \end{bmatrix}$$

are given by $\lambda = -\alpha x \pm \sqrt{\alpha^2 x^2 + \beta}$. Hence at the point $P \equiv (p_1, \ \ p_2)$,

$|\lambda| = \left|-\alpha p_1 \pm \sqrt{\alpha^2 {p_1}^2 + \beta}\right| < 1.$

This proves that the fixed point $P \equiv \left(\frac{1}{2\alpha}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 4\alpha}\right], \ \ \frac{\beta}{2\alpha}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 4\alpha}\right]\right)$ is an attracting fixed point of the Henon map if $\alpha \in \left(\frac{-(1-\beta)^2}{4}, \ \ \frac{3(1-\beta)^2}{4}\right) = I$.

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Similarly, it can be proved that $Q \equiv \left(\frac{1}{2\alpha}\left[\beta - 1 - \sqrt{(1-\beta)^2 + 4\alpha}\right],\ \frac{\beta}{2\alpha}\left[\beta - 1 - \sqrt{(1-\beta)^2 + 4\alpha}\right]\right)$ is a saddle point of the Henon map $f(x,\ y)$.

This completes the proof.

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2.4 Period doubling cascade in the Henon Map

From the above theorem it follows that the points $\frac{-(1-\beta)^2}{4}$ and $\frac{3(1-\beta)^2}{4}$ are bifurcation points for the Henon map, where $0 < \beta < 1$. As the value of the parameter α becomes greater than $\frac{3(1-\beta)^2}{4}$, an attracting 2-cycle is observed for the Henon map. The attracting period 2-cycle can be obtained by solving the equation

$[f \circ f](x,\ y) = (1 - \alpha(1 - \alpha x^2 + y)^2 + \beta x,\ \beta(1 - \alpha x^2 + y))$, which is a fourth degree equation. It can be proved that $f(x,\ y)$ has a period doubling bifurcation at the parameter value $\alpha = \frac{3(1-\beta)^2}{4}$. As the values of the parameter α increases further, the attracting period 4-cycle, period 8-cycle, period 16-cycle, etc are observed. For the fixed value $\beta = 0.4$, a period 1-cycle is observed for $\alpha = 0.2$, a period 2-cycle is observed for $\alpha = 0.5$, a period 4-cycle is observed for $\alpha = 0.9$ and so on. As α becomes greater than 1.01, this period doubling behavior is not observed, but in fact an unpredictability of the iteration values is noted.

2.5 Strange attractor of the Henon Map

For $\alpha = 1.2$ and $\beta = 0.4$, we have plotted the 6000 iterates of the Henon map and obtained the curve as shown in the figure 1.

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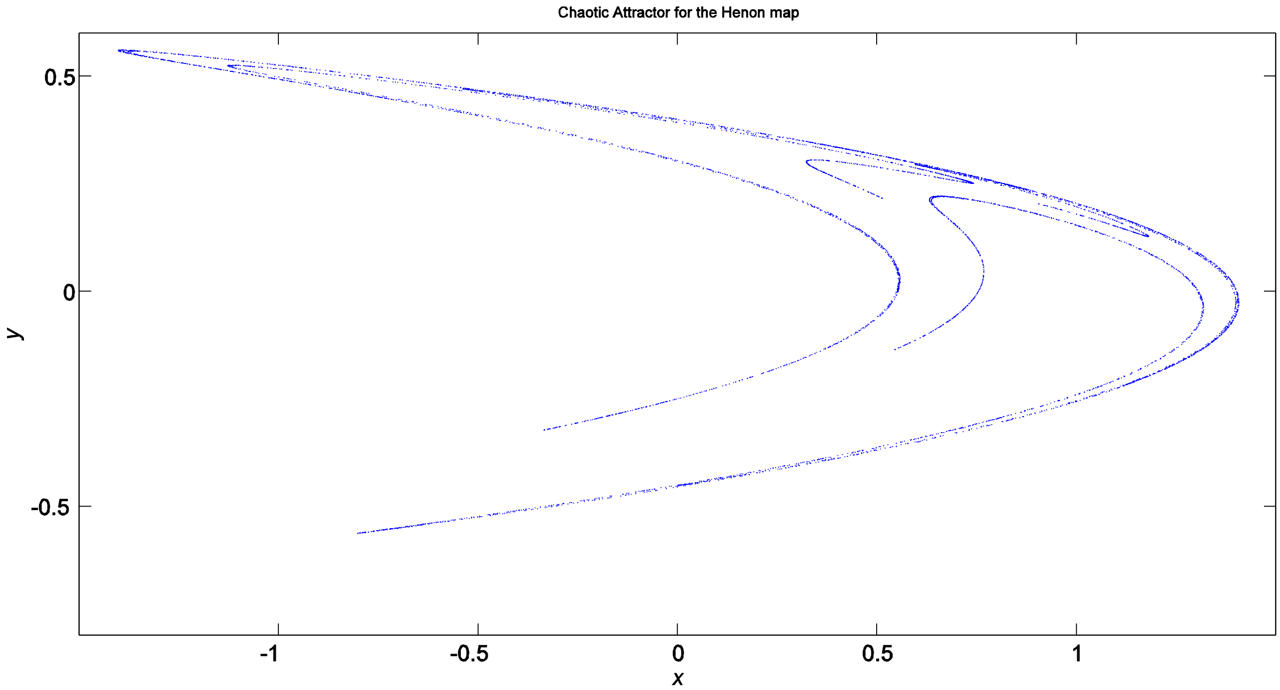

Figure 1

With simple eyes, the curve in the figure 1 appears to be a simple 2-dimensional curve, but what makes it special is that if we zoom in on a small portion of the curve, we see that the curve has the exactly same pattern as the original one. This can be observed from the figure 2 which is the zoom in of the curve in the figure 1.

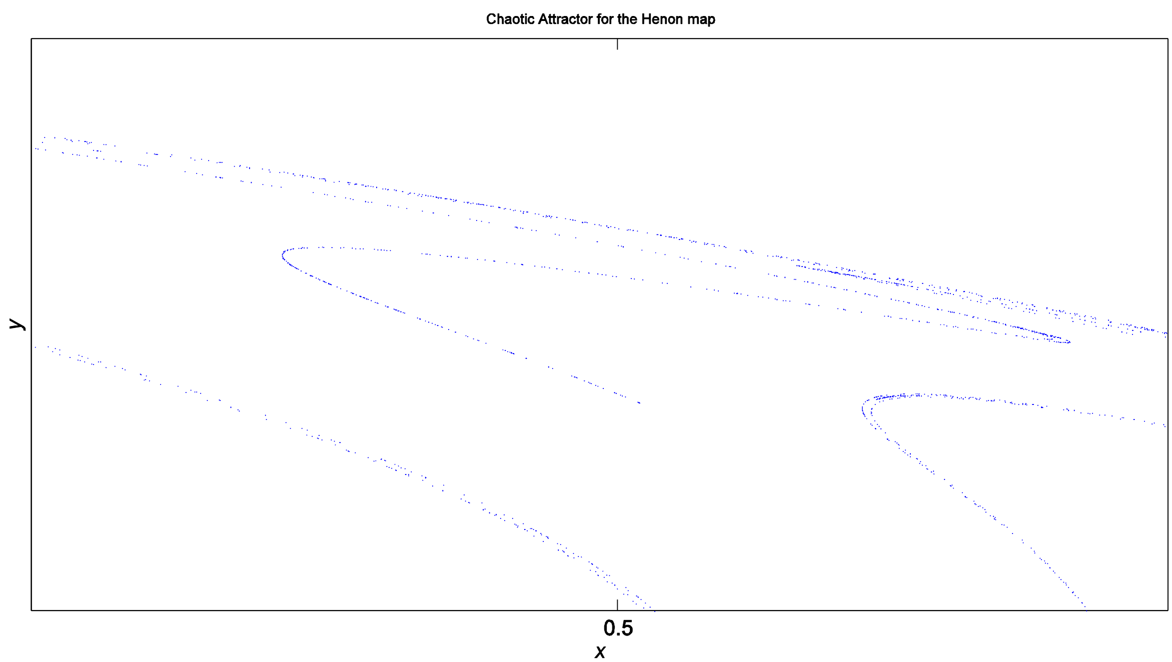

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Figure 2

This proves that the iterates repeat themselves around the attracting fixed point. Thus the curve in the figure is a Cantor like set which has a fractal dimension. Also, the second specialty of the curve is that it has a sensitive dependence on the initial conditions. The curve as shown in the figure 1 is obtained for the initial values $(x_0, y_0) = (0.1, 0)$. The curve with another set of initial conditions $(x_0, y_0) = (0.1, 0.1)$, which is quite close to the first one is as shown in the figure 3.

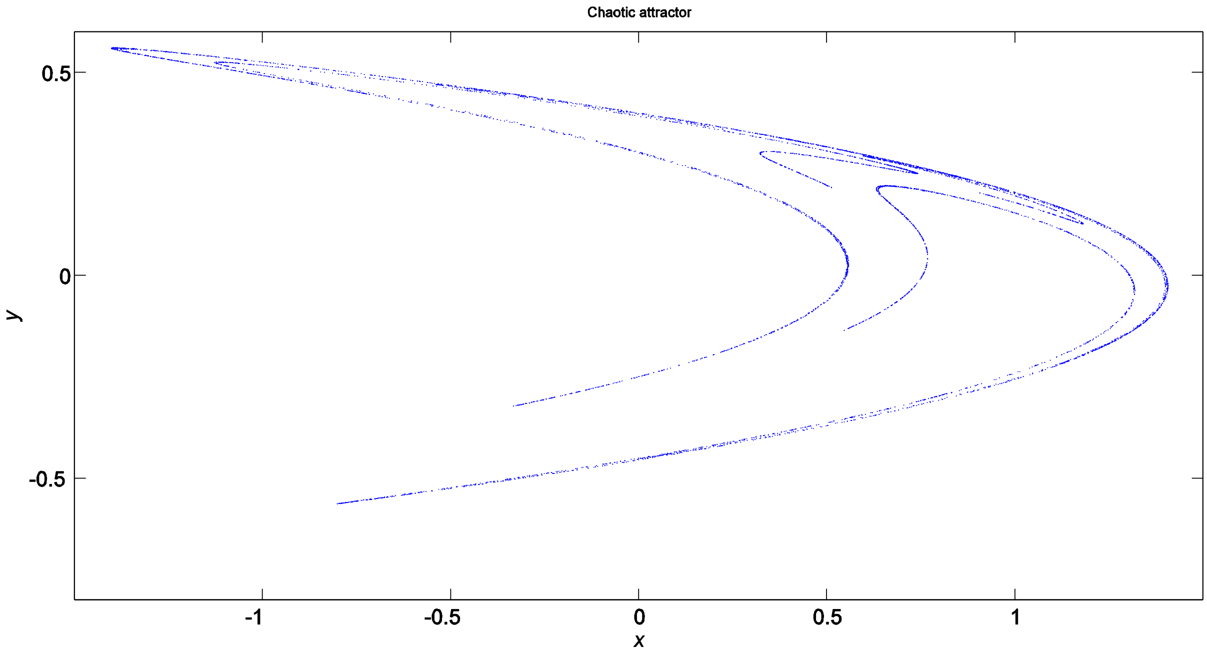

Figure 3

The sensitive dependence on initial conditions can be observed from the figure 1 and figure 3. This type of attractor that has sensitive dependence on the initial conditions is called as the *chaotic attractor* or a *strange attractor* and the mapping or the system in this case is called as the chaotic system.

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