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APPROXIMATED VOLUME OF CONE AND SPHERE FROM DIFFERENT POINT OF VIEW

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Dedication: *I have dedicated this paper to Prof. Rishikesh Ghose and Prof. Dr. Md. Nazrul Haque Chowdhury Sir. They were my favorite and honorable teachers in College life. Both of them were prominent and devoted teachers of the Department of Mathematics. Prof. Rishikesh Ghose was also the teacher of Prof. Dr. Md Nazrul Haque Chowdhury. I salute these two great personalities because they continue to be our source of inspiration, creating deep positive lines in the minds of countless people.*

ABSTRACT
This paper attempts to shed light on how the volume of a cone and sphere can be perceived and derived through approximation from different perspectives. Although we all know the volume of cone and sphere, this is a small effort to get junior math readers out of their superstitions and into a new learning process.
KEYWORDS: ***Solid, Cone, Sphere, Cylinder, Length, Area, Volume***

1. INTRODUCTION
In solid geometry, the cone and sphere are very familiar, important, interesting, and integral part. In general, Cone is a 3D object with a circular base, and the lateral surface of the cone connects its base and apex. Sphere is the only solid geometrical shape having no base and roof, the whole of which is the curved surface. The science world is indebted to the eminent Italian scientist Archimedes (287 BC - 212 BC) of Syracuse for determining the lateral surface area and volume of cone and sphere and a lot of contribution to solid geometry. It is astonishing how he has provided formulae for explaining, analyzing, and accurately measuring these things without the convenience of modern machinery, please see [1, 2]. Geometry is currently presenting to junior mathematics readers in new, easier, and more successful theoretical and demonstrating ways, based on the results given by Archimedes, see [3, 4]. As a modern method, Calculus plays a significant role in determining the volume of cone and sphere, see [5, 6]. Presentation of solid geometry subjects to readers in a display mode has become quite popular now. We may watch the videos [7, 8, 9]. Now we are trying to discuss the methods how to calculate or obtain the volume of cone and sphere.

2. PRELIMINARIES
2.1 Object in various dimensions and spaces: In mathematics, philosophical terms object is both abstract and concrete conception or thing which, measured in 0D/1D/2D/3D space, but in physics, it is a body/matter in 3D bounded space. In geometry, the point is nothing but a conception that is a dimensionless quantity containing the most fundamental object only. It has no part, i.e., it does not occupy an object. It indicates only a position in any 0D, 1D, 2D, 3D,…, space, see [10]. When a point goes on moving, it creates a line in 1D/2D space.
Line segment, ray, straight line, curved line, angle (1D/2D) or area, and volume, can be formed by a combination of points, i.e., all these are the set of points. In other words, points are responsible for an object. A straight line is a one-dimensional object in 1D space, but a curved line is a one-dimensional object in 2D space. One-dimensional plane angle or plane area is a two-dimensional object in 2D space, but a two-dimensional solid angle or curved area is a two-dimensional object in 3D space.

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2.2 Area and Volume in space: Both plane and curved surface area are two-dimensional measurements in 2D and 3D space, respectively. If an angle is measured at a point, subtended by a length/line segment, then through this angle, actually we measure an area. Area means how many spaces can be covered by the surface of a 2D object, whether the area plane or curve. When a base area goes on moving/rotation, it creates a 3D object, i.e., (3D object made by 2D area), see [11]. In general, the volume of a 3D object means how much space can be occupied by the enclosed surface/object in 3D space. The volume of a 3D object is dependent on the amount of base area of that object. If the geometrical shape of the 3D object is regular, then the volume can measure easily by applying the formulae. Often by rearranging the shape of the object, we can also find out the required volume.

2.3 Volume of Tetrahedron: A tetrahedron is a triangular-based pyramid. The volume of a tetrahedron is $V_{tetra} = \frac{1}{3} \times$ Area of base $\times$ Height $= \frac{1}{3} \times \Delta \times h$, where Δ= Base area and h = Height.

2.4 Volume of Cylinder: If r be the radius of base and h be the height of a right circular cylinder, then the volume of the cylinder is $V_{cyl} =$ Area of base $\times$ Height $= \frac{1}{2} \times ঌ \times r^2 \times h = \pi r^2 h$, where ঌ $= 2\pi$ and ঌ $= 6.28$ = Probir's Constant, please see [12]

2.5 Surface area and Volume of Cube: If r be the length of sides of a cube, then the total surface area and volume of the cube are $S_{cube} = 6 \times 2r \times 2r = 24\, r^2$ and $V_{cube} = r \times r \times r = r^3$.

3. RESULTS AND DISCUSSION

3.1 Demonstration – 1; Volume of the sphere through cube

Suppose a sphere with radius r is the inscribed sphere in a cube. So, the cube is the circumscribed cube with sides $2r$ about that sphere which has shown in the figure below:

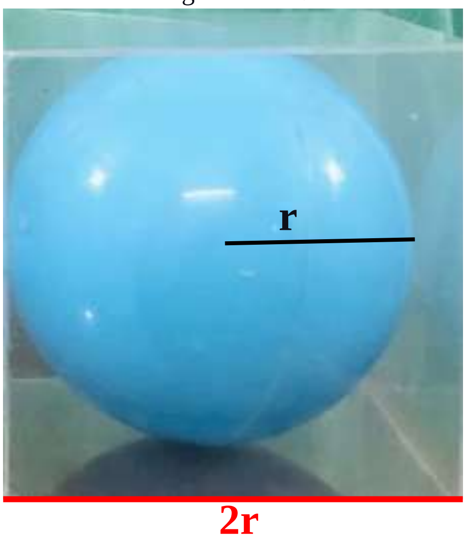

Fig: 1 (Sphere inscribed in a cube)

The ratio of volume and total surface area of the inscribed Sphere in a cube is the same as the ratio of volume and total surface area of the circumscribed Cube about that Sphere.

$\therefore \frac{V_{sphere}}{S_{sphere}} = \frac{V_{cube}}{S_{cube}}$ or $\frac{V_{sphere}}{4\,\pi\, r^2} = \frac{2r \times 2r \times 2r}{6 \times 2r \times 2r}$ or $V_{sphere} = \frac{4}{3}\pi r^3$ cubic units.

3.2 Demonstration – 2; Volume of the sphere through tetrahedron:

Let us consider a sphere whose radius is r. Now we take one-eighth part of the object (sphere) and find the volume of that part which has shown in the following figures (2 – 3):

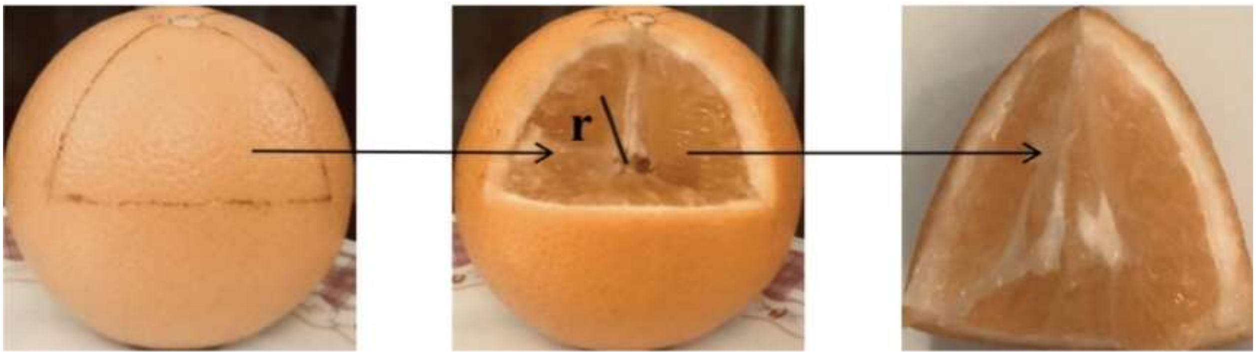

Fig: 2 (One-eighth part of the sphere)

Because for the last stage of fig: 2, the distance of each point on the surface from the apex is r, then we can consider this part of the whole object the same as a tetrahedron by flattening the surface and making the edges of the base into straight line segments, where the surface area remains constant. The next figure has given below:

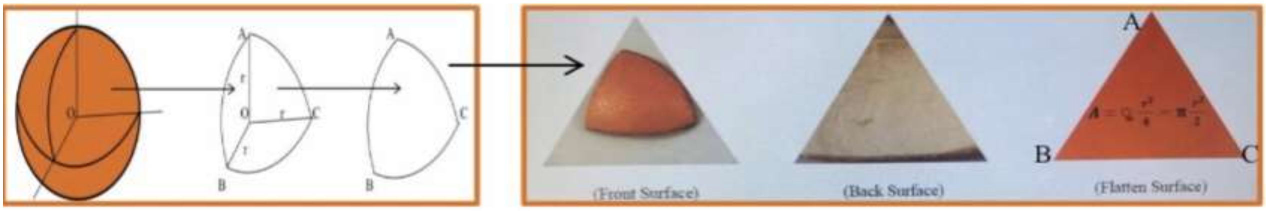

Fig: 3 (Flatten process of the base of the one-eighth part of the sphere)

From the above figures, we have got the base area of the tetrahedron $A = \Delta = \frac{1}{8} \times 2\pi r^2 = \pi \frac{r^2}{4} = \pi \frac{r^2}{2}$, see [4].

Figure relationship gives us the following tetrahedron:

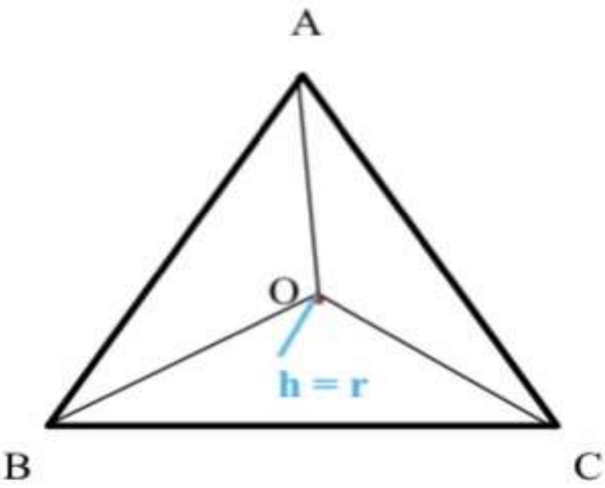

Fig: 4 (One-eighth part of the sphere = Tetrahedron of height $h = r$)

The volume of the tetrahedron is $V_{tetra} = \frac{1}{3} \times \Delta \times h = \frac{1}{3} \times \pi \frac{r^2}{4} \times r = \pi \frac{r^3}{12} = \pi \frac{r^3}{6}$, see sub-section 2.3

Therefore, the required volume of the sphere is $V_{sphere} = 8 \times V_{tetra} = 8 \times \pi \frac{r^3}{12} = \frac{2}{3} \pi r^3 = \frac{4}{3} \pi r^3$ cubic units, see [13].

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3.3 Demonstration – 3; Volume of the sphere through a Cylinder

3.3.1 Converting the Sphere into a Cylinder

Cylinder, Cone, and Sphere are very closely related to one another in the case of volume measurement. Among them, the cylinder's volume is easier to understand and, that is why we are going to determine the volume of the Sphere with the help of the cylinder. For this, we take the figure below:

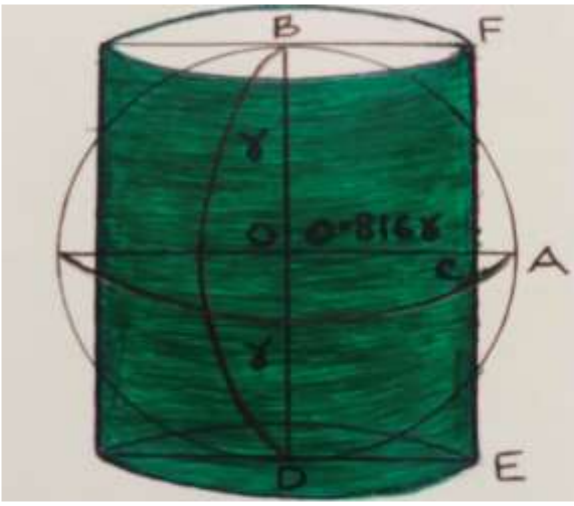

Fig: 5 (Volume of the sphere through a Cylinder)

We apply a procedure in fig: 5 is to make the great-circle sector OAB into rectangle OCFB without changing the height OB = r. Now it is clear that the half-cylinder with base radius r, height r, is created by the single rotation of rectangle OCFB. For the new making cylinder, the scaling is OA = OB = 16 cells = 1 inch = r, OC = 13.056 cells (approximated) = 0.816 inches = 0.816 r. Therefore, the volume of this new half-cylinder is the same as the volume of the hemisphere of radius OA = r, height OB = r because, in a reverse way, we can come back to the hemisphere from the half-cylinder, see [14, 15].

Here, the volume of the green-colored cylinder is $V_{cyl} = 2 \times \frac{1}{2} \pi\, (0.816\ \mathrm{r})^2 \times r = 0.666\, \pi\, r^3 = \frac{2}{3} \pi\, r^3 = \frac{4}{3}\, \pi\, r^3$, see subsection 2.4.

Therefore, the volume of the sphere with radius r, height $h = 2r$ is $V_{sphere} = \frac{2}{3} \pi\, r^3 = \frac{4}{3}\, \pi\, r^3$ cubic units.

3.3.2 Lemma 1: The volume of a sphere with radius r and height $h = 2r$ is two-thirds of the volume of a cylinder with the same base radius r and height $2r$.

Proof: The ratio of volume and total surface area of a sphere is the same as the ratio of volume and total surface area of a cylinder, where radius and height for both are the same. Let the total surface area and volume of the sphere and cylinder be (S_{sphere}, S_{cyl}) and (V_{sphere}, V_{cyl}), where r = radius of sphere = radius of the base of a cylinder and $2r$ be the height of both.

We know that the total surface area of sphere S_{sphere} = Lateral Surface area = $4\pi r^2$, the total surface area of cylinder S_{cyl} = Lateral Surface area + Base area + Roof area = $4\pi r^2 + \pi r^2 + \pi r^2 = 6\pi r^2$.

Now, $\frac{V_{sphere}}{S_{sphere}} = \frac{V_{cyl}}{S_{cyl}}$ or $\frac{V_{sphere}}{4\pi r^2} = \frac{V_{cyl}}{6\pi r^2}$ or $V_{sphere} = \frac{2}{3} V_{cyl}$

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3.3.3 Corollary: The volume of a sphere with a radius of r is $\frac{2}{3}$ ဥ $r^3 = \frac{4}{3}\pi r^3$

Let the volume of a sphere is V_s, where r = Radius of the sphere. Now, from the lemma, we have the volume of sphere $V_{sphere} = \frac{2}{3} V_c$, where V_{cyl} = The volume of a cylinder with radius r and height $2r$. Therefore, the volume of the sphere is $V_{sphere} = \frac{2}{3} \times$ Base $\times$ Height $= \frac{2}{3} \times \frac{1}{2}$ ဥ $r^2 \times 2r = \frac{2}{3}$ ဥ $r^3 = \frac{4}{3}\pi r^3$ cubic units.

3.4 Demonstration – 4; Volume of the sphere through cone

3.4.1 Volume of Right Circular Cone

A right circular cone is a three-dimensional object in 3D space. Geometrically cones can be formed by rotation of right-angled triangle about perpendicular/base(axis) whose whole lateral surface is a curved surface with a circular base. Sometimes it is considered a pyramid with a circular base instead of a polygon because its lateral surface meets at an apex. However, now here we are going to find out the volume of a cone. It is worth noting that the volume of a right circular cone and oblique cone with the same circular base and height are the same.
We consider the figure below:

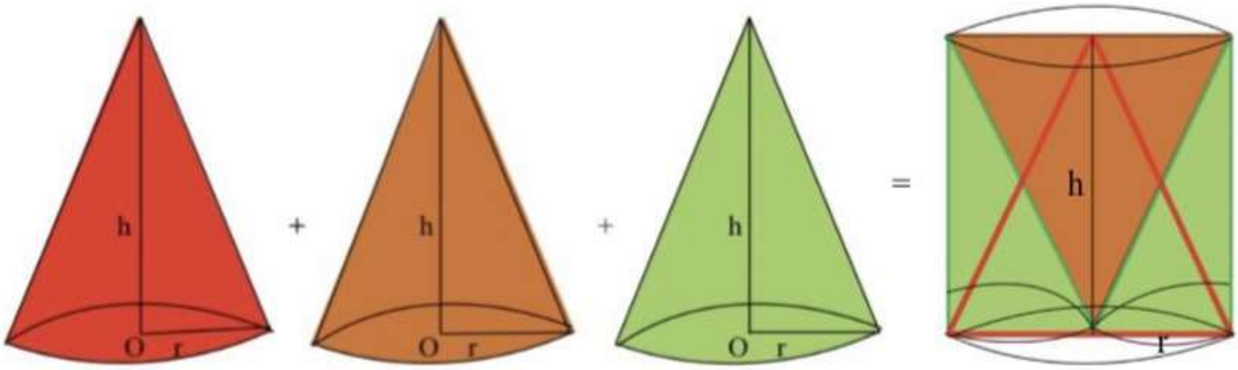

Fig: 6 (Three cones make a cylinder)

Analysis of fig: 6: Suppose the red-colored cone is a hollow right circular cone whose height is h and radius of the base is r. A similar description is applicable for the orange and green-colored cones. Now, let us fix the position of the red-colored cone and connect the orange-colored cone with the red-colored cone by exchanging the locations of the top and center of the base, i.e., in mutually opposite positions. Finally, we cut the green-colored cone through the vertex to its vertical height, then attach the right part of the two equal pieces to the left-blank space of the previous description and the remaining left part to the right-blank space.
From the above demonstration, we can conclude that three same cones make a cylinder with the same radius of the base and the same height, watch [16]. The volume of a cylinder is easy to understand and, it is $V_{cyl} = \frac{1}{2}$ ဥ $r^2h = \pi r^2 h$, see subsection 2.4. Therefore, the volume of a cone with base radius r and height h is $V_{cone} = \frac{1}{3} \times \frac{1}{2}$ ဥ $r^2h = \frac{1}{6}$ ဥ $r^2h = \frac{1}{3}\pi r^2 h$ cubic units. If a cylinder and a cone have the same base radius r and height $2r$, then the volume of the cylinder and the cone will be $V_{cyl} =$ ဥ $r^3 = 2\pi r^3$ and $V_{cone} = \frac{1}{3}$ ဥ $r^3 = \frac{2}{3}\pi r^3$.

Note: Here, one thing is remarkable for fig: 6, we will have to consider space covered by the object in a cylinder, not how much mass is contained by the cylinder. If we encircle the model connected by three same cones with a surface, the cylinder with radius r and height h will be visible.

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3.4.2 Volume of a sphere through cone

Let us consider the following object like a sphere (Orange) and its various stages:

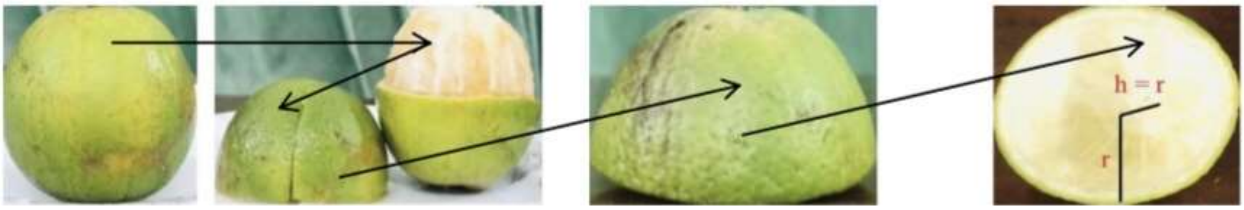

Fig: 7 (Pick up the hollow hemisphere from the whole orange)

Here, we take the hollow hemisphere from an orange whose base is a circle with radius r and height $h = r$. If we flatten the hemisphere, it will become a plane surface that is the lateral surface of a cone and, shown in figure 8.

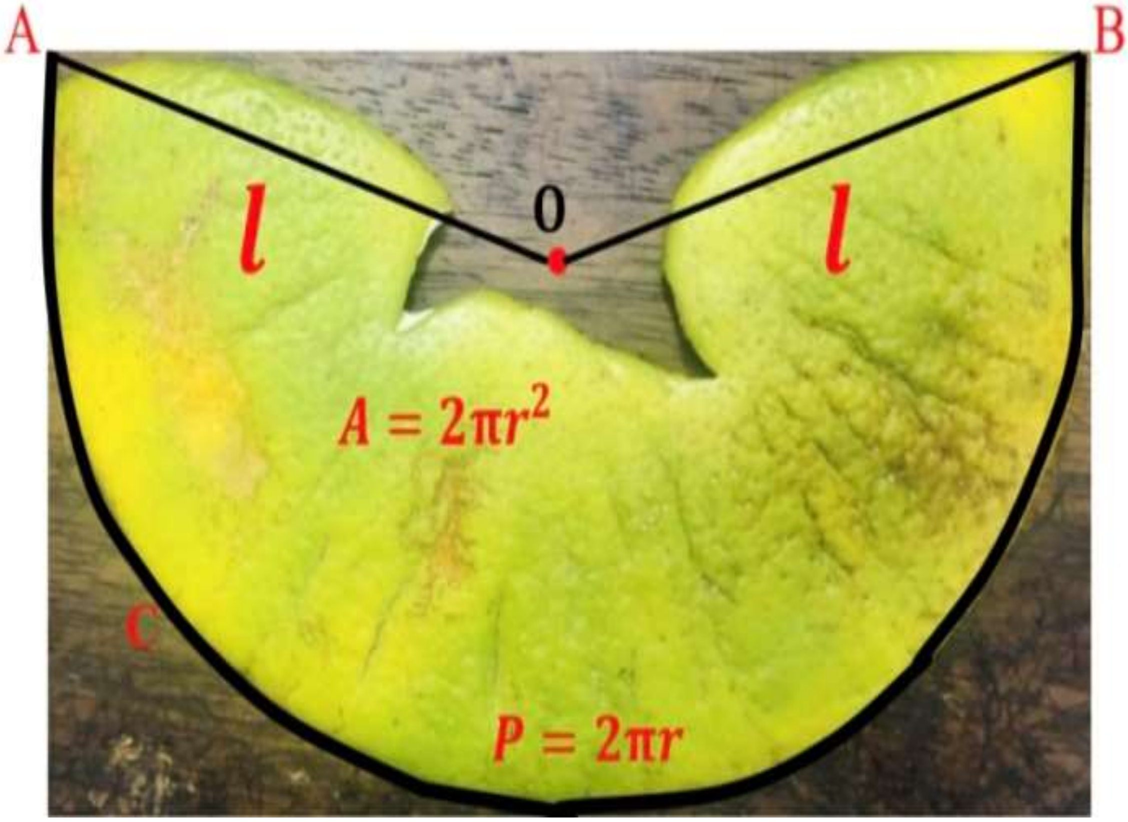

Fig: 8 (Flattened surface of the hemisphere)

Now we are trying to make a cone by the surface of the hemisphere in the next figure. Here, the scale for figure 8 is 16 cells = 1 inch, the radius of the base of sphere and cone r = 17.76 cells = 1.11 inches, OA = OB = l = 39.68 cells = 2.48 inches (approximated) = 2.23 r = Slant height of cone, perimeter ACB = P = 2πr = 7 inches. Therefore the cone becomes:

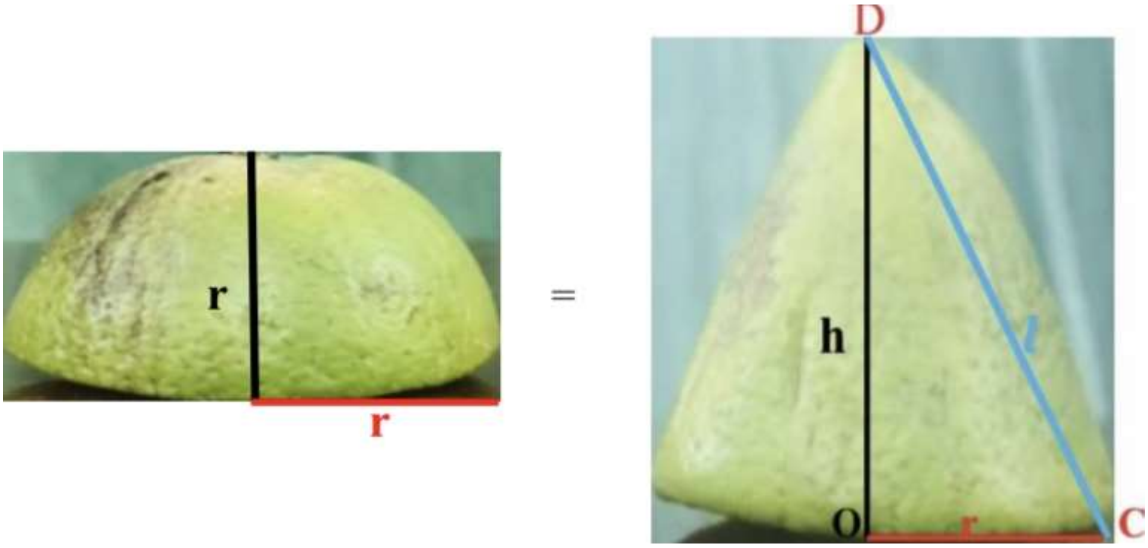

Fig: 9 (Making cone from the hemisphere)

In figure 9, it is seen that the hemisphere with a base radius $r =$ 1.11 and height $r =$ 1.11 becomes a cone with a base radius OC $= r =$ 1.11 and height OD $= h =$ 35.53 cells = 2.22 = $2r$.

The figure relationship is as follows:

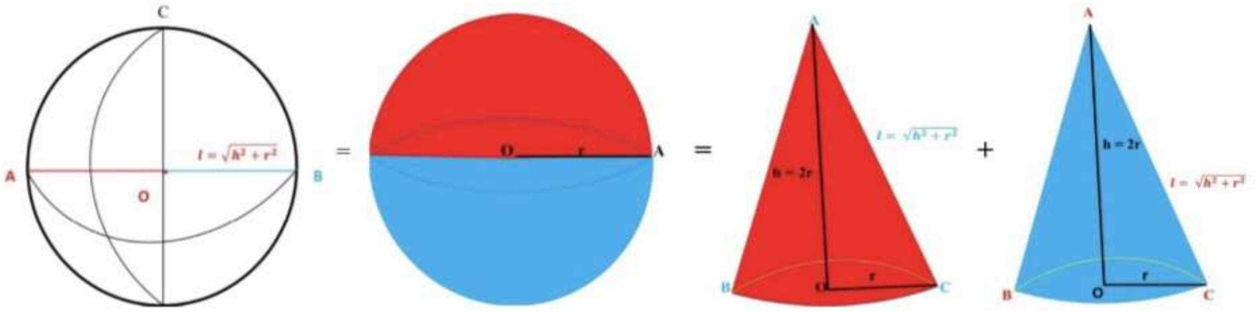

Fig: 10 (Two cones make a sphere)

From the above relationship, we can say that the volume of a sphere is twice that the volume of a cone with the same base radius and height of the sphere, i.e., $V_{sphere} = 2 \times \frac{1}{3} \times \frac{1}{2} \pi r^2 \times h = 2 \times \frac{1}{3} \times \frac{1}{2} \pi r^2 \times 2r = \frac{2}{3} \pi r^3 = \frac{4}{3} \pi r^3$ cubic units.

3.5 Volume of a sphere from an algebraic relationship

From fig: 6 and 10, we have got $3\, Cone = 1\, Cylinder$ and $2\, Cone = 1\, Sphere$ $\therefore$ $3\, Cone = \frac{3}{2}\, Sphere$. Therefore, $\frac{3}{2}\, Sphere = 1\, Cylinder$ or $Sphere = \frac{2}{3}\, Cylinder$. So, the volume of a sphere with radius r is $V_{sphere} = \frac{2}{3} \times$ Volume of cylinder with base radius r and height $2\,r = \frac{2}{3} \times \frac{1}{2} \pi r^2 \times 2r = \frac{2}{3} \pi r^3 = \frac{4}{3}\pi r^3$ cubic units.

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3.6 Demonstration – 5; Volume of the sphere through the inscribed and circumscribed sphere and cube

3.6.1 Volume of a frustum of a cone: If R, r, and h be the radius of the lower circular area, the radius of the upper circular area, and height of a frustum of a cone, then the volume of the frustum = $\frac{\pi h}{3}(R^2 + R r + r^2)$ cubic units.

3.6.2 Inscribed and Circumscribed sphere and cube: Suppose a sphere S_I with radius r is the inscribed sphere in a cube C_I, So, the length of the sides of the cube C_I will be $2r$, and this cube is a circumscribed cube about the sphere S_I. Again, let S_O be the circumscribed sphere of the cube C_I as well as the inscribed sphere in another cube C_O, So, the length of the sides of the cube C_O will be $2\sqrt{2}\, r$, and this cube is the circumscribed cube about the sphere S_O which shown in the figure below:

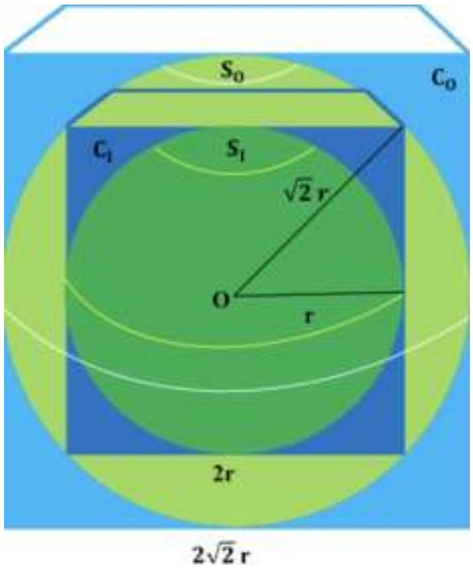

Fig: 11 (Inscribed and Circumscribed Sphere and Cube)

3.6.3 Lemma 2: The volume of the circumscribed sphere S_O of a cube C_I is the $2\sqrt{2}$ multiple of the volume of the inscribed sphere S_I in that cube C_I, please see [12] for the similar.

Proof: Let S_I be the inscribed sphere with radius r in a cube C_I, and the volume of this sphere is $V_I = V$. Here, it is clear that the length of the sides of the cube C_I will be $2r$. Again, let S_O be the circumscribed sphere of that cube C_I. So the radius of the circumscribed sphere is $\sqrt{2}\, r$, and the length of the sides of the circumscribed cube C_O about the sphere S_O will be $2\sqrt{2}\, r$. Suppose the volume of the sphere S_O is V_O. Now, the ratio of the volume of the circumscribed and inscribed sphere S_O, S_I of a cube C_I is the same as the ratio of the circumscribed and inscribed cube C_O, C_I about the sphere S_O.

Therefore, $\frac{V_O}{V_I = V} = \frac{\text{Volume of the cube } C_O}{\text{Volume of the cube } C_I} = \frac{(2\sqrt{2}\, r)^3}{(2\ r)^3} = \frac{16\sqrt{2}}{8} = 2\sqrt{2} \therefore\ V_O = 2\sqrt{2}\, V.$

Now, we take the following horizontal/vertical cross-section towards the center of the above-mentioned 3D model in figure 11 and pick up the inscribed hemisphere from the circumscribed hemisphere, where a hole creates in the circumscribed hemisphere. Then finally, we convert this circumscribed hemisphere into a frustum of a cone which, is shown in the following figure:

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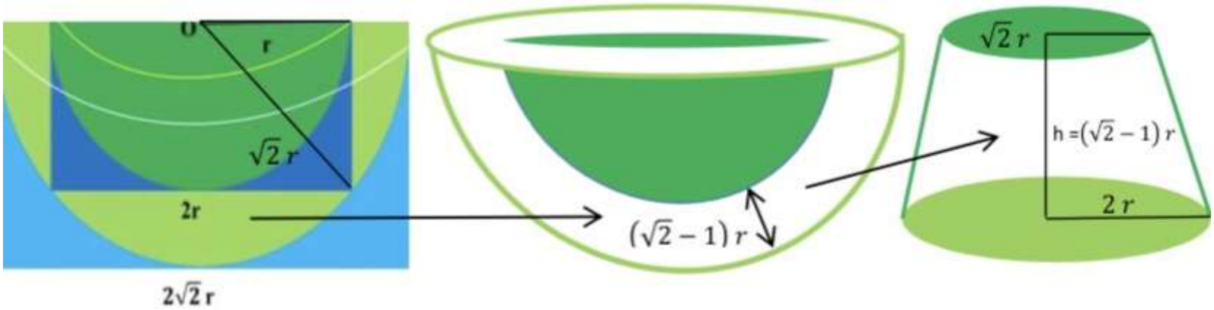

Fig: 12 (Frustum of a cone from the 3D model of figure 11)

In figure 12, the inscribed hemisphere has picked from the circumscribed hemisphere and, as a result, the inner surface area of the bowl liked object is = $2\pi r^2 = \pi\,(\sqrt{2}\,r)^2$, the outer surface area of the bowl is = $2\pi(\sqrt{2}\,r)^2 = \pi\,(2\,r)^2$, the thickness of the bowl is = $\sqrt{2}\,r - r = (\sqrt{2}-1)r$. Now, the bowl can be converted into a frustum of a cone whose radius of the upper circular area is = $\sqrt{2}r$ (from the surface area), the radius of the lower circular area is = $2r$, and height is $h = (\sqrt{2}-1)r$. So the volume of the frustum of the cone is = $\frac{\pi(\sqrt{2}-1)\,r}{3}\,\{(2r)^2 + 2r \times \sqrt{2}\,r + (\sqrt{2}\,r)^2\} = \frac{\pi(\sqrt{2}-1)\,r}{3}\,(6+2\sqrt{2}\,)r^2 = 1.22\,\pi\,r^3$, see 3.6.1

Finally, we have an algebraic and geometric relationship that the difference between volumes of the circumscribed and inscribed sphere is twice the volume of the above frustum of a cone.

Therefore, $V_O - V_I = 2 \times 1.22\,\pi\,r^3$ or $2\sqrt{2}\,V - V = 2.44\,\pi\,r^3$, see 3.6.3 or $(2\sqrt{2}-1)\,V = 2.44\,\pi\,r^3$ or $1.83\,V = 2.44\,\pi\,r^3$ or $V = \frac{2.44}{1.83}\pi\,r^3 = (1.3333\,\ldots\ldots)\,\pi\,r^3$(approximated), i.e., the required volume of the sphere with radius r is $V = (1.3333\,\ldots\ldots)\pi\,r^3 \approx \frac{4}{3}\,\pi\,r^3 = \frac{2}{3}$ ◌ r^3 cubic units.

4. CONCLUSION

There are many beautiful methods for determining the volume of a sphere that is understandable to pioneer students but often complicated for new readers or those at the primary level regarding the sphere's volume. In particular, care has been taken in this paper to build credibility in the formula before memorizing the subject among junior readers. We think that the volume of a sphere has been discussed more times before in a much better way than the method we have discussed, but in our described methods, it has been discussed here, through demonstrative materials. We hope that it will play a positive role in developing the practice of formulation among new readers. Different alternative methods have been demonstrated here in this attempt to find the volume because the adaptive power of different readers is several. Moreover, the alternative method enhances the beauty of mathematics and strengthens the foundation of belief in the formula. We expect mathematicians will try to discuss the volume of the sphere in a simple, more beautiful, and alternative way, and we will assume that the small effort has been successful if our discussed method has made the slightest contribution to the readers.

Acknowledgment

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