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OBSERVATION ON THE PAPER ENTITLED INTEGRAL SOLUTION OF THE HOMOGENEOUS TERNARY CUBIC EQUATION $x^3+y^3=52(x+y)z^2$

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ABSTRACT

This paper concerns with the problem of obtaining non zero distinct integer solutions to the homogeneous ternary cubic equation $x^3+y^3=52(x+y)z^2$. ***Also, formulae for generating sequence of integer solutions based on the given solution are presented.***

KEYWORDS: ***Ternary cubic, Integer solutions, Homogeneous cubic, Generation of solutions.***

INTRODUCTION

The theory of Diophantine equations in multidegree with multivariables offers a rich variety of interesting and fascinating problems[1-4]. One may refer [5-22] for cubic equation with three variables. It is observed that in [22]the authors have presented some patterns of integer solutions to the ternary cubic equation $x^3+y^3=52(x+y)z^2$. It is noted that the above equation has other choices of non-zero distinct integer solutions.

Thus, in this paper, the other choices of non-zero distinct integer solutions to the above ternary cubic equation are obtained. Also, formulas for generating sequence of integer solutions based on the given solution are presented.

METHODS OF ANALYSIS

The homogeneous ternary cubic equation under consideration is

$$x^3+y^3=52(x+y)z^2 \tag{1}$$

To start with , it is observed that (1) is satisfied by the triples

(x,,y,z) = $(16,12,\pm 2)$,$(6,-2, \pm 1)$,$(16,4, \pm 2)$,$(48,36, \pm 6)$.

However, we have other sets of nonzero distinct integer solutions to (1) which are illustrated below.

Introduction of the linear transformations

x=2(u+v),y=2(u-v) (2)

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in (1) , it is written as

$$u^2 + 3v^2 = 13z^2 \tag{3}$$

The above equation is solved through different method for obtaining the values of

u,v,z. Substituting the values of u,v in (2) the corresponding values of x and y satisfying (1) are found.

We present below different methods of solving (3) and in view of (2), one obtains different sets of integer solutions to (1).

Set 1 :

Assume z as

$$z = a^2 + 3b^2 \tag{4}$$

Write 13 on the R.H.S. of (3) as

$$13 = (1+i2\sqrt{3})(1-i2\sqrt{3}) \tag{5}$$

Using (4) & (5) in (3) and employing the method of factorization, consider

$$u + i\sqrt{3}v = \left(1 + i2\sqrt{3}\right)\left(a + i\sqrt{3}b\right)^2$$

After Equating the real and imaginary terms on both sides , it is seen that

$$u = a^2 - 3b^2 - 12ab$$

$$v = 2a^2 - 6b^2 + 2ab$$

Using in (2), one has

$$\left.\begin{aligned} x &= 2(3a^2 - 9b^2 - 10ab) \\ y &= 2(-a^2 + 3b^2 - 14ab) \end{aligned}\right\} \tag{6}$$

Note : 1

In addition to (5), the integer 13 on the R.H.S. of (3) is written as

$$13 = \frac{\left(7 + i\sqrt{3}\right)\left(7 - i\sqrt{3}\right)}{4} \quad \text{or}$$

$$13 = \frac{\left(5 + i3\sqrt{3}\right)\left(5 - i3\sqrt{3}\right)}{4}$$

Following the above procedure ,one may obtain different set of integer solutions to (1).

Set 2 :

(3) is written as

$$u^2 = 13z^2 - 3v^2 = u^2 * 1 \qquad (7)$$

Assume u as

$$u = 13a^2 - 3b^2 \qquad (8)$$

Write 1 on the R.H.S. of (7) as

$$1 = \left(\sqrt{13} + 2\sqrt{3}\right)\left(\sqrt{13} - 2\sqrt{3}\right) \qquad (9)$$

Using (8) and (9) in (7) and employing the method of factorisation, consider

$$\sqrt{13}z + \sqrt{3}v = \left(\sqrt{13} + 2\sqrt{3}\right)\left(\sqrt{13}a + \sqrt{3}b\right)^2 \qquad (10)$$

Equating the corresponding parts, one has

$z = 13a^2 + 3b^2 + 12ab$, $v = 26a^2 + 6b^2 + 26ab$

Therefore, in view of (2), the corresponding integer solutions to (1) are given by

$x = 2(39a^2 + 3b^2 + 26ab)$, $y = 2(-13a^2 - 9b^2 - 26ab)$

Note : 2

In addition to (9), the integer 1 on the R.H.S. of (7) is written as

$$1 = \frac{\left(2\sqrt{13} + \sqrt{3}\right)\left(2\sqrt{13} - \sqrt{3}\right)}{49} \text{ or}$$

$$1 = \frac{\left(2\sqrt{13} + 3\sqrt{3}\right)\left(2\sqrt{13} - 3\sqrt{3}\right)}{25}$$

Following the above procedure, one may obtain different set of integer solutions to (1).

Set 3:

(3) is written as

$$3v^2 = 13z^2 - u^2 \qquad (11)$$

Assume v as

$$v = 13a^2 - b^2 \qquad (12)$$

Write the integer 3 on the L.H.S. of (11) as

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$$3 = \left(2\sqrt{13}+7\right)\left(2\sqrt{13-7}\right) \tag{13}$$

Using (12), (13) in (11) and employing the method of factorisation, consider

$$\sqrt{13}z + u = \left(2\sqrt{13}+7\right)\left(\sqrt{13}a+b\right)^2 \tag{14}$$

Equating the rational and irrational parts,

z= $26a^2 + 2b^2 + 14ab$

u= $91a^2 + 7b^2 + 52ab$

In view of (2), the corresponding integer solutions to (1) are given by

x= $2\left(104a^2 + 6b^2 + 52ab\right)$

y= $2\left(78a^2 + 8b^2 + 52ab\right)$

Note : 3

In addition to (13), the integer 3 on the L.H.S. of (11) is written as

$$3 = \frac{\left(2\sqrt{13}+5\right)\left(2\sqrt{13}-5\right)}{9}$$

Following the above procedure, one may obtain different set of integer solution to (1).

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solutions are presented below:

Let $\left(u_0, v_0, z_0\right)$ be any given solutions to (3)

Formula 1:

Let $\left(u_1, v_1, z_1\right)$ given by

$$u_1 = h - 2u_0, v_1 = h - 2v_0, z_1 = 2z_0 \tag{15}$$

be the second solution to (1). Using (15) in (1) and simplifying, one obtains

h= $u_0 + 3v_0$

In view of (17), the values of $u_1 and\ v_1$ are written in the matrix form as

$$\left(u_1, v_1\right)^t = M\left(u_0, v_0\right)^t$$

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where

M= $\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

and t is the transpose.

The repetition of the above process leads to the n^{th} solutions u_n, v_n given by

$$(u_n, v_n)^t = M^n (u_0, v_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 2, \beta = -2$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I), I = 2 \times 2 \ Identity\ Matrix$$

Thus , the general formulas for integer solutions to (1) are given by

$$z_n = 2^n z_0,$$

$$x_n = 2(u_n + v_n) = 2\left[u_0\left(\frac{2\alpha^n + 2\beta^n}{4}\right) + v_0\left(\frac{6\alpha^n - 2\beta^n}{4}\right)\right]$$

$$y_n = 2(u_n - v_n) = 2\left[\beta^n u_0 - \beta^n v_0\right]$$

Formula 2:

Let (u_1, v_1, z_1) given by

$$u_1 = u_0, v_1 = v_0 + 2h, z_1 = h - z_0 \qquad (16)$$

be the second solution to (1). Using (18) in (1) and simplifying, one obtains

h= $12v_0 + 26z_0$

In view of (17), the values of $v_1 and\ z_1$ are written in the matrix form as

$$(v_1, z_1)^t = M(v_0, z_0)^t$$

where

M= $\begin{bmatrix} 25 & 52 \\ 24 & 51 \end{bmatrix}$

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and t is the transpose.

The repetition of the above process leads to the n^{th} solutions v_n, z_n given by

$$(v_n, z_n)^t = M^n (v_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 38 + \sqrt{1417}, \ \beta = 38 - \sqrt{1417}$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I), I = 2 \times 2 \ Identity \ Matrix$$

Thus , the general formulas for integer solutions to (1) are given by

$$u_n = u_0$$

$$v_n = v_0 \left\{ \frac{\alpha^n + \beta^n}{2} - \frac{13(\alpha^n - \beta^n)}{2\sqrt{1417}} \right\} + \frac{26}{\sqrt{1417}}(\alpha^n - \beta^n) z_0$$

$$z_n = \frac{12}{\sqrt{1417}}(\alpha^n - \beta^n) v_0 + \left\{ \frac{\alpha^n + \beta^n}{2} + \frac{13(\alpha^n - \beta^n)}{2\sqrt{1417}} \right\} z_0$$

$$x_n = 2(u_n + v_n)$$

$$y_n = 2(u_n - v_n)$$

Formula 3:

Let (u_1, v_1, z_1) given by

$$u_1 = 4h - 3u_0, v_1 = 3v_0, z_1 = 3z_0 + h \tag{17}$$

be the second solution to (1). Using (19) in (1) and simplifying, one obtains

h= $8u_0 + 26z_0$

In view of (17), the values of $u_1 and\ z_1$ are written in the matrix form as

$$(u_1, z_1)^t = M(u_0, z_0)^t$$

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where

$$M=\begin{bmatrix} 29 & 104 \\ 8 & 29 \end{bmatrix}$$

and t is the transpose.

The repetition of the above process leads to the n^{th} solutions u_n, z_n given by

$$(u_n, z_n)^t = M^n (u_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 29 + 8\sqrt{13}, \beta = 29 - 8\sqrt{13}$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I), I = 2 \times 2 \ Identity\ Matrix$$

Thus , the general formulas for integer solutions to (1) are given by

$$u_n = u_0\left(\frac{\alpha^n + \beta^n}{2}\right) + \frac{\sqrt{13}}{2} z_0 (\alpha^n - \beta^n)$$

$$z_n = \frac{1}{2\sqrt{13}} u_0 (\alpha^n - \beta^n) + z_0 \left(\frac{\alpha^n + \beta^n}{2}\right)$$

$$x_n = 2(u_n + v_n)$$

$$y_n = 2(u_n - v_n)$$

CONCLUSION

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic Diophantine equation $x^3 + y^3 = 52(x+y)z^2$ representing homogeneous cone. As there are varieties of cones, the readers may search for other forms of cones to obtain integer solutions for the corresponding cones.

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