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ON C* GENERALIZED η-CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT
The aim of this paper is to introduce the notion of c* generalized η-closed sets in topological spaces and study their basic properties. It is the weaker form of closed and generalized c*-closed sets. Further, we shall see that the collection of c* generalized η-closed sets is not closed under finite intersection but it is closed under arbitrary union. Also, we establish the relationship between this new class of closed sets with other existing classes of generalized closed sets in general topology.

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KEYWORDS: ***η-open, c*-open, gc*-closed, rgη-closed, c*gη-closed.***

1. INTRODUCTION

In 1937, Stone [7] introduced the notion of regular open sets. In 1965, Njastad [**6**] introduced the concept of α-open sets. In 1968, the notion of π-open sets were introduced by Zaitsev [**10**] which are weak form of regular open sets. In 1970, Levine [**4**] initiated the study of generalized closed (briefly g-closed) sets. In 2000, Dontchev and Noiri [**2**] introduced the notion of πg-closed sets. Sundaram and John [**8**] introduced the notion of w-closed sets. In 2017, Malathi and Nithyanantha Jothi [5] introduced the concepts of c*-open and generalized c*-closed sets in topological spaces. In 2019, Subbulakshmi, Sumathi and Indirani [**9**] introduced and investigated the notion of η-open sets. In 2021, Kumar [**3**] introduced the concepts of rgη-closed sets and obtained some basic properties of rgη-closed sets.

2. PRELIMINARIES

Throughout this paper, spaces (X, $\Im$), (Y, $\wp$), and (Z, σ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** [**7**] (resp. **regular closed**) if A $\subset$ int(cl(A)) (resp. A $\subset$ cl(int(A)). The finite union of regular open sets is said to be **π-open [10]**. The complement of a π-open set is said to be **π-closed**.

Definition 2.1. A subset A of a topological space (X, $\Im$) is said to be

(i) **α-open** [**6**] if A $\subset$ int(cl(int(A))).

(ii) **η-open** [**9**] if A $\subset$ in(cl(int(A))) $\cup$ cl(int(A)).

(iii) **η-closed** [**9**] if A $\supset$ cl(int(cl(A))) $\cup$ int(cl(A)).

(iv) **c*-open** [**5**] if in(cl(A)) $\subset$ A $\subset$ cl(int(A)).

The complement of a α-open (resp. η-open, c*-open) set is called **α-closed** (resp. **η-closed, c*-closed**). The intersection of all α-closed (resp. η-closed) sets containing A, is called **α-closure** (resp. **η-closure**) of A, and is denoted by α-**cl(A)** (resp. **η-cl(A))**. The **η-**

interior of A, denoted by **η-int(A)** is defined as union of all η-open sets contained in A. We denote the family of all η-open (resp. η-closed) sets of a topological space by **η-O(X)** (resp. **η-C(X)**).

Definition 2.2. A subset A of a space (X, ℑ) is said to be
(1) **generalized closed** (briefly **g-closed**) [**4**] if cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(2) **πg-closed** [**2**] if cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(3) **generalized η-closed** (briefly **gη-closed**) [**3**] if η-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(4) r**gη-closed** [**3**] if η-cl(A) ⊂ U whenever A ⊂U and U is regular-open in X.
(5) **w-closed** [**8**] if cl(A) ⊂ U whenever A ⊂U and U is semi-open in X.
(6) **generalized c*-closed** [**5**] (briefly **gc*-closed**) if cl(A) ⊂ U whenever A ⊂ U and U is c*-open in X.
(7) **g-open** (resp**. πg-open**, **gη-open, rgη-open, w-open, gc*-open**) set if the complement of A is g-closed (resp. πg-closed, gη-closed, rgη-closed, w-closed, gc*-closed).

3. C* GENERALIZED η-CLOSED SETS

In this section, we introduce c* generalized η-closed sets in topological spaces. Also, we derive some of their basic properties of c* generalized η-closed sets.

Definition 3.1. A subset A of a topological space X is said to be c* generalized η-closed (briefly, c*gη-closed) if η-cl(A) ⊂ H whenever A ⊂ H and H is c*-open in X.

Example 3.2. Let X = {a, b, c, d} with topology ℑ = {ϕ, {a}, {c}, {d}, {a, c}, {a, d}, {c, d}, {a, c, d}, X}. Then the subsets ϕ, {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, X are c*gη-closed.

Proposition 3.3. Let X be a topological space. Then every closed set is c*gη-closed.

Proof. Let A be a closed set. Then A = cl(A). Let H be a c*-open set containing A. Then cl(A) ⊂ H. Since η-cl(A) ⊂ cl(A), we have η-cl(A) ⊂ H. Therefore, A is c*gη-closed.

The converse of Proposition 3.3 is not true. This can be proved by the following example.

Example 3.4. Let X = {a, b, c, d} with topology ℑ = {ϕ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then the subset A = {a, b} is c*gη-closed but not closed.

Proposition 3.5. Let X be a topological space. Then every η-closed set is c*gη-closed.

Proof. Let A be a η-closed set. Then A = η-cl(A). Let H be a c*-open set containing A. Then η-cl(A) ⊂ H. Therefore, A is c*gη-closed.

The following example shows that the converse of Proposition 3.5 need not be true in general.

Example 3.6. In Example 3.4, A = {a, b} is c*gη-closed but not η-closed.

Proposition 3.7. Let X be a topological space. Then every π-closed set is c*gη-closed.

Proof. Let A be a π-closed set. Since every π-closed set is closed, we have A is closed. Therefore, by Proposition 3.3, A is c*gη-closed.

The converse of Proposition 3.7 need not be true which can be verified from the following example.

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Example 3.8. In Example 3. 4, A = {a, b} is c*gη-closed but not π-closed.

Proposition 3.9. Let X be a topological space. Then every regular-closed set is c*gη-closed.

Proof. Let A be a regular-closed set. Since every regular-closed set is closed, we have A is closed. Therefore, by Proposition 3.3, A is c*gη-closed.

The converse of Proposition 3.9 need not be true as seen from the following example.

Example 3.10. In Example 3.4, A = {a, b} is c*gη-closed but not regular-closed.

Proposition 3.11. Let X be a topological space. Then every w-closed set is c*gη-closed.

Proof. Let A be a w-closed set. Let H be a c*-open set containing A. Since every c*-open set is semi-open, we have H is semi-open. By our assumption, cl(A) ⊂ H. Since η-cl(A) ⊂ cl(A), we have η-cl(A) ⊂ H. Therefore, A is c*gη-closed.

The converse of Proposition 3.11 need not be true as shown in the following example.

Example 3.12. In Example 3.4, A = {a, b} is c*gη-closed but not w-closed.

Proposition 3.13. Let X be a topological space. Then every gc*-closed set is c*gη-closed.

Proof. Let A be a gc*-closed set. Let H be a c*-open set containing A. Then cl(A) ⊂ H. Since η-cl(A) ⊂ cl(A), we have η-cl(A) ⊂ H. Hence A is c*gη-closed.

The following example shows that the converse of Proposition 3.13 need not be true in general.

Example 3.14. Let X = {a, b, c, d, e} with topology $\mathfrak{I}$ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, c, d}, {a, b, c, e}, X}. Then the subset A = {c, d} is c*gη-closed but not gc*-closed.

Proposition 3.15. Let X be a discrete topological space. Then every πg-closed set is c*gη-closed.

Proof. Let A be a πg-closed set in X. Let H be a c*-open set containing A. Since X is a discrete space, we have H is open in X. By Proposition 3.15 [**1**], H is π-open. This implies, cl(A) ⊂ H. Since η-cl(A) ⊂ cl(A), we have η-cl(A) ⊂ H. Therefore, A is c*gη-closed.

The g-closed and c*gη-closed sets are independent. For example, in Example 3.16, the subset A = {a, b} is g-closed but not c*gη-closed and the subset B = {a, d, e} is c*gη-closed but not g-closed.

Proposition 3.16. Let X be a topological space. Then every c*gη-closed set is rgη-closed.

Proof. Let A be a c*gη-closed set. Let H be a regular-open set containing A. Since every regular-open set is c*-open, so H is c*-open set. Since A is c*gη-closed set, so η-cl(A) ⊂ H. Therefore, A is rgη-closed.

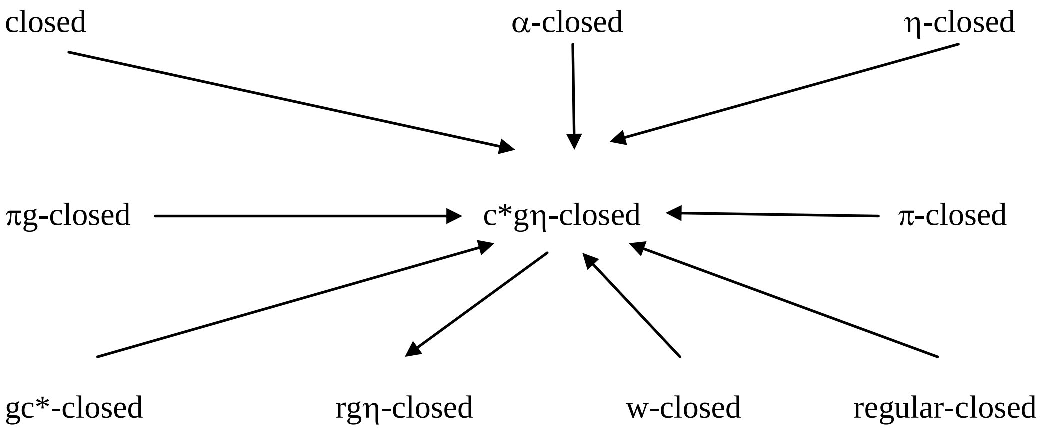
The converse of Proposition 3.11 need not be true.

Proposition 3.17. Let X be a topological space. Then every α-closed set is c*gη-closed.

Proof. Let A be a α-closed set. Let H be a c*-open set containing A. Since A is α-closed set, so A = α-cl(A) ⊂ H. Since η-cl(A) ⊂ α-cl(A) ⊂ H, we have η-cl(A) ⊂ H. Therefore, A is c*gη-closed.

The converse of Proposition 3.11 need not be true.

Remark 3.18. From the above definitions and known results the relationship between c*gη-closed sets and some other existing generalized closed sets are implemented in the following Figure:



Where none of the implications is reversible as can be seen from the above examples:

4. SOME BASIC PROPERTIES OF C*Gη-CLOSED SETS

Proposition 4.1. In a topological space X, arbitrary union of c*gη-closed sets is c*gη-closed.

Proof. Let A_1, A_2,…,A_n,… be c*gη-closed subsets of X. Let $A = \cup \{A_i : i \in I\}$. Let H be a c*-open set containing A. Then each $A_i \subset H$. Since each A_i is c*gη-closed, we have $\eta\text{-cl}(A_i) \subset H$. This implies $\cup \{\eta\text{-cl}(A_i) : i \in I\} \subset H$. This implies, $\eta\text{-cl}(\cup \{A_i : i \in I\}) \subset H$. That is, η-cl(A) ⊂ H. Therefore, A is c*gη-closed.

The intersection of two c*gη-closed subsets of a space X need not be c*gη-closed. For example, let X = {a, b, c, d, e} with topology $\Im$ = {ϕ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then A = {a, c} and B = {a, d} are c*gη-closed sets but their intersection A ∩ B = {a} is not c*gη-closed set.

Proposition 4.2 If a subset A of a space X is c*gη-closed set in X, then η-cl(A) − A does not contain any non-empty c*-open set in X.

Proof. Assume that A is c*gη-closed set in X. Suppose H is a c*-open set such that H ⊂ η-cl(A) − A and H ≠ ϕ. Then H ⊂ X − A. This implies, A ⊂ X − H. Since H is a c*-open, we have X − H is also a c*-closed set in X. Then η-cl(A) ⊂ X − H. This implies, H ⊂ X − η-cl(A). Also, H ⊂ η-cl(A). Hence H ⊂ cl(A) ∩ (X − η-cl(A)) = ϕ, which contradicts H ≠ ϕ. Hence η-cl(A) − A does not contain any non-empty c*-open set in X.

Proposition 4.3. Let X be a topological space. Then for any element p ∈ X, the set X − {p} is either c*gη-closed or c*-open.

Proof. Suppose X − {p} is not a c*-open set. Then X is the only c*-open set containing X − {p}. This implies, η-cl(X − {p}) ⊂ X. Hence X − {p} is a c*gη-closed set in X.

The following Proposition gives the necessary and sufficient condition for a c*gη-closed to be η-closed.

Proposition 4.4. Let A be a c*gη-closed set in a space X. Then A is η-closed if and only if η-cl(A) – A is c*-open.

Proof. Suppose A is η-closed. Then η-cl(A) = A. This implies, η-cl(A) – A = ϕ, which is c*-open.

Conversely, suppose η-cl(A) – A is a c*-open set in X. By Proposition 4.2, η-cl(A) – A = ϕ. This implies, A = η-cl(A). Hence A is η-closed.

Proposition 4.5. Let X be a topological space. If A is a c*gη-closed subset of X such that A ⊂ B ⊂ η-cl(A), then B is a c*gη-closed set in X.

Proof. Let H be a c*-open set containing B. Then A ⊂ H. Since A is c*gη-closed, we have η-cl(A) ⊂ H. This implies, η-cl(B) ⊂ H. Hence B is c*gη-closed set in X.

Proposition 4.6. Let X be a topological space. If X and ϕ are the only c*-open sets then all the subsets of X are c*gη-closed.

Proof. Let A be a subset of X. If A = ϕ, then A is c*gη-closed. If A ≠ ϕ, then X is the only c*-open set containing A. This implies η-cl(A) ⊂ X. Hence A is c*gη-closed.

The converse of the Proposition 4.6 is not true. This can be proved by the following example.

Example 4.7. Let X = {a, b, c, d} with topology ℑ = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {c, d}, {a, b, c}, {b, c, d}, {a, c, d}, X}. Then all the subsets of X are c*gη-closed. But the c*-open sets are ϕ, {a}, {b}, {a, b}, {c, d}, {a, c, d}, {b, c, d}, X.

Next Proposition gives the characterization for c*gη-closed.

Proposition 4.8. A subset A of a space X is c*gη-closed if and only if for each A ⊂ H and H is c*-open, there exists a η-closed set F such that A ⊂ F ⊂ H.

Proof. Suppose A is a c*gη-closed set and A ⊂ H and H is c*-open. Then η-cl(A) ⊂ H. If we put F = η-cl(A), then F is η-closed and A ⊂ F ⊂ H.

Conversely, assume that H is a c*-open set containing A. Then there exists a η-closed set F such that A ⊂ F ⊂ H. Since η-cl(A) is the smallest η-closed set containing A, we have A ⊂ η-cl(A) ⊂ F. Also, since F ⊂ H, we have η-cl(A) ⊂ H. Hence A is c*gη-closed.

Proposition 4.9. If A is η-closed and B is c*gη-closed subset of a space X, then A ∪ B is c*gη-closed.

Proof. Let H be a c*-open set containing A ∪ B. Then A ⊂ H and B ⊂ H. Since B is c*gη-closed and B ⊂ H, we have η-cl(B) ⊂ H. Then A ∪ B ⊂ A ∪ η-cl(B) ⊂ H. Since A is η-closed, we have A ∪ η-cl(B) is η-closed. Hence there exists a η-closed set A ∪ η-cl(B) such that A ∪ B ⊂ A ∪ η-cl(B) ⊂ H. Therefore, by Proposition 4.8, A ∪ B is c*gη-closed.

Proposition 4.10. If A is closed and B is c*gη-closed subset of a space X, then A ∪ B is c*gη-closed.

Proof. Since every closed set is η-closed, we have A is η-closed. Also, by Proposition 4.9, B is c*gη-closed. Therefore, by Proposition 4.9, A ∪ B is c*gη-closed.

Proposition 4.11. If a subset A of a topological space X is c*gη-closed, then η-cl(A) – A does not contain any nonempty regular-open (resp. regular-closed) set in X.

Proof. Suppose H is a regular open set contained in η-cl(A) – A and H ≠ ϕ. Since every regular open set (resp. regular-closed set) is c*-open, we have H is c*-open. Thus, H is a c*-open set contained in η-cl(A) – A. Therefore, by Proposition 4.2, H = ϕ. This is a contradiction. Therefore, η-cl(A) – A does not contain any non-empty regular-open (resp. regular-closed) set in X.

Already we proved that every η-closed set is c*gη-closed but the converse is not true in general (see Proposition 3.5). The following Proposition shows that when the reverse implication is true.

Proposition 4.12 If a subset A of a topological space X is both c*-open and c*gη-closed, then A is η-closed.

Proof. Suppose A is both c*-open and c*gη-closed. Then η-cl(A) ⊂ A. Also, A ⊂ η-cl(A). This implies, A = η-cl(A). Therefore, A is η-closed.

Proposition 4.13. Let X be a topological space and A be a subset of X. If A is regular-open and c*gη-closed, then A is both η-open and η-closed.

Proof. Assume that A is regular-open and c*gη-closed. Since every regular-open set is c*-open, by Proposition 4.12, A is η-closed. Since regular-open set is η-open, we have A is η-open. Thus, A is both η-open and η-closed.

CONCLUSION

In this paper, we have introduced c*gη-closed sets in topological spaces and studied some of their basic properties. Also, we have studied the relationship between c*gη-closed sets with some other existing generalized closed sets in topological spaces. The c*gη-closed set can be used to derive a new decomposition of closed map, open map, continuity, homeomorphism, and new separation axioms. This idea can be extended to topological ordered spaces, bitopological spaces, bitopological ordered spaces and fuzzy topological spaces.

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