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A STUDY ON THE KERR SOLUTION

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ABSTRACT
In this study, a novel form of the Kerr solution is presented. The solution comprises a time coordinate which signifies the local appropriate time for the free-falling viewers on a set of the simple trajectories. Various physical events are chiefly clear when associated to this time coordinate. The selected coordinates also confirm that the solution is well performed at the horizon. The solution is well fit to the tetrad formalism and a suitable null tetrad is existing. Also given the Dirac Hamiltonian and, for one choice of the tetrad, it contains on a simple, Hermitian form.

KEYWORDS : ***Kerr solution, Dirac Hamiltonian, tetrad, Minkowski spacetime***

I. INTRODUCTION

The Kerr solution is very important in astrophysics forever since it was realized that the accretion processes would incline to spin up a black hole to the near its critical rotation rate. Several forms of the Kerr solution at this time exist in the literature. Most of these are confined in Chandrasekhar's work, and beneficial summaries are delimited in the books by D' Inverno and Kramer et al. The purpose of this study is to present a new form of the Kerr solution which has even now proved to be valuable in numerical simulations. It is a direct extension of the known Schwarzschild solution

$$ds^2 = dt^2 - \left(dr + \left(\frac{2M}{r}\right)^{\frac{1}{2}} dt\right)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (1)$$

This is obtained from the Eddington Finkelstein form

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\bar{t}^2 - \frac{4M}{r} d\bar{t}dr - \left(1 + \frac{2M}{r}\right) dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (2)$$

by the coordinate transformation

$$t = \bar{t} + 2(2Mr)^{\frac{1}{2}} - 4M \ln\left(1 + \left(\frac{r}{2M}\right)^{\frac{1}{2}}\right) \qquad (3)$$

In both the equations, $0 < r < \infty, \theta$ and ϕ take their usual meaning.

The equation (1) has several nice features, many of which outspread to the Kerr case. The Kerr solution is good at the horizon, so it can be employed safely to analyze physical processes nearby the horizon, and indeed inside it. Alternative beneficial feature is that the time t overlaps with the appropriate time of viewers free-falling end to end radial trajectories initial from rest at infinity. This is probable since the velocity vector

$$\dot{x}^i = \left(1, -\left(\frac{2M}{r}\right)^{\frac{1}{2}}, 0, 0\right) \qquad \dot{x}_i = (1, 0, 0, 0) \qquad (4)$$

expresses a radial geodesic with the constant θ and ϕ. The appropriate time along these paths overlaps with t, and the geodesic eq^n is basically

$$\ddot{r} = \frac{-M}{r^2} \qquad (5)$$

Physics as realized by these viewers is practically completely Newtonian, making this gauge a very advantageous one for presenting some of the more problematic thoughts of the black hole physics. The numerous gauge selections foremost to this form of the "Schwarzschild solution" as well carry over in the existence of the matter and provide a meek system for the study of the creation of spherically symmetric clusters and the black holes.

Additional beneficial feature of the time coordinate in equation (1) is that it allows the "Dirac equation" in a Schwarzschild background to be cast in an unassuming Hamiltonian form. The full Dirac equation is gotten by totaling a single term $\widehat{H}_I$ to the "free-particle" Hamiltonian in the form Minkowski spacetime. This added term is

$$\widehat{H}_I\psi = i\left(\frac{2M}{r}\right)^{\frac{1}{2}}(\partial_r\psi + 3/(4r)\psi) = i\left(\frac{2M}{r}\right)^{\frac{1}{2}} r^{\frac{-3}{4}}\,\partial_r\left(r^{\frac{3}{4}}\,\psi\right) \tag{6}$$

A convenient feature of this gauge is that, the measure on surfaces of the constant t is the identical as that of the "Minkowski spacetime," so one can work standard methods from the quantum theory with a tiny modification. One refinement is that the Hamiltonian is not the self-adjoint because of the existence of the uniqueness. This reveals the situation as a decay in the wave function as the current density is sucked onto the singularity.

The time coordinate t in equation (1) has various of the properties of a global, 'Newtonian time'. This recommends that an attempt to find a correspondent for the "Kerr solution" might fail by reason of its the angular momentum. The key to accepting how to complete a suitable generalization is the realization, that it is one the local properties of time t that make it so suitable for recitation the physics of the solution. The usual extension for the "Kerr Solution" is consequently to look for an appropriate set of reference viewers which generalizes the idea of a family of the viewers on radial trajectories. In the Sections two and three discuss an innovative form of the "Kerr Solution" and show that it has several of the wanted properties. In Section four several tetrad forms of the solution, and give a Hermitian form of the Dirac Hamiltonian. All over using the Latin letters for the spacetime indices and Greek letters for the tetrad indices, and use the signature $\eta_{\alpha\beta} = diag\,(+\;---)$. Natural units $c = G = \bar{h} = 1$ are working throughout.

II. THE KERR SOLUTION

The novel form of the "Kerr Solution" can be written in Cartesian coordinates (t, x, y, z) in the Kerr-Schild form. In this coordinate system our novel form of the solution is

$$ds^2 = \eta_{ij}dx^i dx^j - \left(\frac{2\alpha}{\rho}a_i v_j + \alpha^2 v_i v_j\right)dx^i dx^j \tag{7}$$

where η_{ij} is the Minkowski metric,

$$\alpha = \frac{(2Mr)^{\frac{1}{2}}}{\rho} \tag{8}$$

$$\rho^2 = r^2 + \frac{a^2 z^2}{r^2} \tag{9}$$

here a and M are constants. The function r is specified implicitly by

$$r^4 - r^2(x^2 + y^2 + z^2 - a^2) - a^2 z^2 = 0 \tag{10}$$

and we restrict r to $0 < r < \infty$, with $r = 0$ relating the disk $z = 0$, $x^2 + y^2 \leq a^2$. The maximally extended "Kerr Solution" (where r is allowed to take $-ve$ values) will not be counted here.

The two vectors in the equation (7) are

$$v_i = \left(1, \frac{ay}{a^2+r^2}, \frac{-ax}{a^2+r^2}, 0\right) \tag{11}$$

and

$$a_i = (r^2 + a^2)^{\frac{1}{2}}\left(0, \frac{rx}{a^2+r^2}, \frac{ry}{a^2+r^2}, \frac{z}{r}\right) \tag{12}$$

These 2 vectors play a vital role in studying physics in a Kerr related. They are linked to the two major null directions $n_{\pm}$ by

$$n_{\pm} = (r^2 + a^2)^{\frac{1}{2}}\, v_i \pm (\alpha\rho v_i + a_i) \tag{13}$$

For computations it is convenient to note that, the contravariant components of the spacelike vector in the brackets are similar as those of $-a_i$,

$$\alpha\rho v^i + a^i = -(r^2+a^2)^{\frac{1}{2}}\left(0, \frac{rx}{a^2+r^2}, \frac{ry}{a^2+r^2}, \frac{z}{r}\right) \tag{14}$$

The vector v_i similarly plays a vital role in unravelling the Dirac equation in a Kerr background, and it is the time like eigenvectors of the form electromagnetic stress-energy tensor for the method "Kerr-Newman analogue" of our form.

III. SPHEROIDAL COORDINATES

From (7) is more visibly exposed if we make known to oblate spheroidal coordinates (r, θ, ϕ), where

$$\cos\theta = \frac{z}{r} \qquad 0 \le \theta \le \pi \tag{15}$$

$$\tan\phi = \frac{y}{x} \qquad 0 \le \phi < 2\pi \tag{16}$$

so that ρ improves its standard definition

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{17}$$

Here use of the symbols r and θ are standard, nevertheless one must be conscious that when $M = 0$ (flat space) these decrease to the oblate spheroidal coordinates, and not the spherical form of polar coordinates. Clearly, from the fact that r does not equal $\sqrt{(x^2 + y^2 + z^2)}$.

In terms of the coordinates (t, r, θ, ϕ) our novel form of the "Kerr solution" is

$$ds^2 = dt^2 - \left(\frac{\rho}{(r^2+a^2)^{\frac{1}{2}}}dr + \alpha(dt - a\sin^2\theta\ d\phi)\right)^2$$
$$-\rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta\, d\phi^2 \tag{18}$$

This effortlessly generalizes the Schwarzschild form of the equation (1), replacing $\sqrt{(2M/r)}$ with $\sqrt{(2Mr)/\rho}$, and presenting a revolving component. It can be simplified more by acquaint with the hyperbolic coordinate η via $asinh\eta = r$, however this can make some equations rigid to interpret and will not be working here. The equation (18) is gotten from the innovative "Eddington-Finkelstein Form" of the "Kerr Solution",

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right)dv^2 - 2dvdr + \frac{2Mr}{\rho^2}(2a\sin^2\theta)dvd\bar{\phi} + 2a\sin^2\theta\, drd\bar{\phi} - \rho^2 d\theta^2 -$$
$$\left((r^2+a^2)\sin^2\theta + \frac{2Mr}{\rho^2}(a^2\sin^4\theta)\right)d\bar{\phi}^2 \tag{19}$$

via the coordinate transformation

$$dt = dv - \frac{dr}{1+(2Mr/(r^2+a^2))^{1/2}} \tag{20}$$

$$d\phi = d\bar{\phi} - \frac{adr}{r^2+a^2+\left(2Mr(r^2+a^2)\right)^{1/2}} \tag{21}$$

This transformation is exact for all r, however the integrals tangled do not give the idea to have a simple closed form.

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The velocity vectors

$$\dot{x}^i = \left(1, -\frac{\alpha(r^2+a^2)^{1/2}}{\rho}, 0, 0\right) \qquad \dot{x}_i = (1, 0, 0, 0) \tag{22}$$

describes an infalling geodesic with constant θ and ϕ, and the zero velocity at infinity. The presence of these 'geodesics' is a key property of the solution. The time coordinate t now has the modest clarification of recording the local appropriate time for spectators in free-fall along trajectories of constant θ and ϕ. As per the spherical case, various physical phenomena are modest to interpret when uttered in terms of this time coordinate. An instance of this is if in the next section, where we display that the time coordinate tends a 'Dirac Hamiltonian' which is the Hermitian in form. The variance between this free-fall velocity and the velocity v_i correspondingly delivers a local meaning of the angular velocity confined in the gravitational field.

IV. TETRADS AND THE DIRAC EQUATION

From the equation (18) and (13) one can construct the following null tetrad, stated in (t, r, θ, ϕ) coordinates,

$$l^i = \frac{1}{r^2+a^2}\left(r^2 + a^2, r^2 + a^2 - \left(2Mr(r^2 + a^2)\right)^{1/2}, 0, a\right) \tag{23}$$

$$n^i = \frac{1}{2\rho^2}\left(r^2 + a^2, -(r^2 + a^2) - \left(2Mr(r^2 + a^2)\right)^{1/2}, 0, a\right) \tag{24}$$

$$m^i = \frac{1}{\sqrt{2}(r+ia\cos\theta)}(ia\sin\theta, 0, 1, i\csc\theta) \tag{25}$$

In this way the Weyl scalars Ψ_0, Ψ_1, Ψ_3 and Ψ_4 all be wiped out, and

$$\Psi_2 = -\frac{M}{(r-ia\cos\theta)^3} \tag{26}$$

A 2nd tetrad, better suitable to computations of the matter geodesics, is given by

$$\begin{aligned} e^0{}_i &= (1, 0, 0, 0) \\ e^1{}_i &= \left(\alpha, \rho/(r^2 + a^2)^{1/2}, 0, -\alpha a \sin^2\theta\right) \\ e^2{}_i &= (0, 0, \rho, 0) \\ e^3{}_i &= \left(0, 0, 0, (r^2 + a^2)^{1/2}\sin\theta\right) \end{aligned} \tag{27}$$

This describes a frame for all that values of the coordinate r, so it is in force inside and outside the horizon. An additional tetrad is on condition that by reverting to the original Cartesian coordinates of equation (7) and writing

$$e^\mu{}_i = \delta_i^\mu - \frac{\alpha}{\rho}v_i a_j \eta^{j\mu} \tag{28}$$

where v_i and a_i are as clear at equations (11) and (12). Now the inverse is

$$e_\mu{}^i = \delta_\mu{}^i + \frac{\alpha}{\rho}\eta^{ij}a_j\delta_\mu{}^k v_k \tag{29}$$

This final form of the tetrad is the meekest to use when using in the Dirac equation in a "Kerr" background. We will not go over the facts here but will just show the final equation in the usual Hamiltonian form. Following the conventions of Itzykson and Zuber we signify the Dirac Pauli matrix illustration of the Dirac algebra by $\{\gamma^\mu\}$ and write $\alpha^i = \gamma^0\gamma^i, i = 1\ldots3$. Since $e_\mu{}^0 = \delta_\mu{}^0$, premultiplying the Dirac equation by γ_0 the Dirac equation in a "Kerr" background becomes

$$i\partial_t\psi = -i\alpha^i\partial_i\psi + m\gamma_0\psi + \hat{H}_{K\psi} \tag{30}$$

where

$$\widehat{H}_{K\psi} = \frac{\sqrt{2M}}{\rho^2}\begin{pmatrix} (r^3 + a^2 r)^{1/4} i\partial_r\left((r^3 + a^2 r)^{1/4}\psi\right) \\ -a\cos\theta\, r^{\frac{1}{4}}\alpha_\phi i\partial_r\left(r^{\frac{1}{4}}\psi\right) - \frac{a\cos\theta}{2}(r^2 + a^2)^{1/2}\gamma_5\psi \end{pmatrix} \quad (31)$$

and

$$\alpha_\phi = -\sin\phi\,\alpha_1 + \cos\phi\alpha_2 \quad (32)$$

Hypersurfaces measure of constant t is again the similar as that of the Minkowski spacetime, As the covariant volume element is basically

$$dxdydz = \rho^2 \sin\theta\, drd\theta d\phi \quad (33)$$

As per the Schwarzschild case the relations Hamiltonian $\widehat{H}_K$ is not independent when integrated concluded these hypersurfaces. This is because the singularity grounds a boundary term to be existing once the Hamiltonian is integrated.

V. CONCLUSIONS

The Kerr solution is very important in astrophysics by means of ever more captivating evidence points to the presence of black holes revolving at nearby their critical rate. In the least form of the solution which helps physical understanding of revolving black holes is evidently beneficial. The form of the solution existing here has several features which realize this aim. The solution is well-matched for studying progressions nearby the horizon, and the condensed form of the spin linking for the tetrad of the eq^n (28) makes it principally good for the numerical computation. It might also be well-known that this gauge confesses a simple generalization to a time-dependent system which looks compatible to the study of accretion and the formation of revolving black holes. The Complete description of the features of this gauge, together with the derivation of the Dirac Hamiltonian will be existing elsewhere. One reason for not prominence more of the advantages here is that various of the hypothetical uses which activity these properties have been executed utilizing the Hestenes' spacetime algebra. This language completely disclosures much of the difficult algebraic structure of the Kerr solution and, brings with it several insights. These are hard to define deprived of employing spacetime algebra and so it will be presented untainted in a distinct paper. The fact that the time coordinate t measured by observers brings the Dirac equation into the Hamiltonian form is expressive of a cavernous principle. These equations also authorizations many methods from quantum field theory to be accepted over to a gravitational related with little change. The lack of independents due to the basis itself is also usual in this framework, as the singularity is an accepted sink for the current. In the non-rotating case, the physical progressions resulting from the existence of this sink are quite meek to analyze. The Kerr case is significantly more intricate, due both to the nature of the fields confidential the internal horizon, and to the edifice of the singularity. One exciting point to note is that the sink area is designated by $r = 0$, and so represents a disk, instead of just a ring of the matter.

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