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(1, 2)*-GENERALIZED η-CLOSED SETS IN BITOPOLOGICAL SPACES

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ABSTRACT
In this paper, we introduce (1, 2)*-generalized η-closed sets and obtain the relationships among some existing closed sets like (1, 2)*-semi- closed, (1, 2)*-α- closed and (1, 2)*-η- closed sets and their generalizations. Also we study some basic properties of (1, 2)*-gη-open sets. Further, we introduce (1, 2)*-gη-neighbourhood and discuss some properties of (1, 2)*-gη-neighbourhood.

1. Introduction

The study of bitopological spaces was first intiated by Kelly [**4**] in the year 1963. By using the topological notions, namely, semi-open, α-open and pre-open sets, many new bitopological sets are defined and studied by many topologists. In 2008, Ravi et al. [**6**] studied the notion of (1, 2)*-sets in bitopological spaces. In 2004, Ravi and Thivagar [**5**] studied the concept of stronger from of (1, 2)*-quatient mapping in bitopological spaces and introduced the concepts of (1, 2)*-semi-open and (1, 2)*-α-open sets in bitopological spaces. Recently H. Kumar [**3**] introduced the concept of (1, 2)*-η-open sets and discuss their properties.

2. Preliminaries

Throughout the paper $(X, \Im_1, \Im_2)$, (Y, σ_1, σ_2) and $(Z, \wp_1, \wp_2)$ (or simply X, Y and Z) denote bitopological spaces.

Definition 2.1. Let S be a subset of X. Then S is said to be **$\Im_{1,2}$-open [5]** if $S = A \cup B$ where $A \in \Im_1$ and $B \in \Im_2$. The complement of a $\Im_{1,2}$-open set is $\Im_{1,2}$-closed.

Definition 2.2 [**5**]. Let S be a subset of X. Then
(i) the $\Im_{1,2}$-closure of S, denoted by $\Im_{1,2}$-cl(S), is defined as $\cap$ {F : S $\subset$ F and F is $\Im_{1,2}$-closed}; (ii) the $\Im_{1,2}$-interior of S, denoted by $\Im_{1,2}$-int(S), is defined as $\cup$ {F : F $\subset$ S and F is $\Im_{1,2}$-open}.

Note 2.3 [**5**]. Notice that $\Im_{1,2}$-open sets need not necessarily form a topology.

Remark 2.4. [**6**]
(i) $\Im_{1,2}$-int(S) is $\Im_{1,2}$-open for each $S \subset X$ and $\Im_{1,2}$-cl(S) is $\Im_{1,2}$ -closed for each $S \subset X$.
(ii) A subset $S \subset X$ is $\Im_{1,2}$-open iff S = $\Im_{1,2}$-int(S) and $\Im_{1,2}$-closed iff S = $\Im_{1,2}$-cl(S).
(iii) $\Im_{1,2}$-int(S) = $\Im_1$-int(S) $\cup$ $\Im_2$-int(S) and $\Im_{1,2}$-cl(S) = $\Im_1$-cl(S) $\cup$ $\Im_2$-cl(S) for any $S \subset X$.
(iv) for any family $\{S_i / i \in I\}$of subsets of X, we have

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(1) $\cup_i \Im_{1,2}$-int(S_i) $\subset \Im_{1,2}$-int($\cup_i S_i$).
(2) $\cup_i \Im_{1,2}$-cl(S_i) $\subset \Im_{1,2}$-cl($\cup_i S_i$).
(3) $\Im_{1,2}$-int($\cup_i S_i$) $\subset \cup_i S_i \Im_{1,2}$-int($S_i$).
(4) $\Im_{1,2}$-cl($\cup_i S_i$) $\subset \cup_i \Im_{1,2}$-cl(S_i).

Definition 2.5. A subset A of a bitopological space (X, $\Im_1$, $\Im_2$) is called
(i) **(1 , 2)*-semi-open** [5] if A = $\Im_{1,2}$-cl($\Im_{1,2}$-int(A)),
(ii) **(1 , 2)*-α-open** [5] if A $\subset \Im_{1,2}$-int ($\Im_{1,2}$-cl($\Im_{1,2}$-int (A))).
(iii) **(1, 2)*-η-open** [5] if A $\subset \Im_{1,2}$ -int($\Im_{1,2}$ -cl($\Im_{1,2}$ -int)(A)) $\cup\ \Im_{1,2}$ -cl($\Im_{1,2}$ -int)(A)).

The complement of a (1, 2)*-semi-open (resp. (1, 2)*-α-open, (1, 2)*-η-open) set is called **(1, 2)*-semi-closed** (resp. **(1, 2)*-α-closed, (1, 2)*-η-closed**).

The **(1, 2)*-semi-closure** (resp. **(1, 2)*-α-closure, (1, 2)*-η-closure**) of a subset A of X is denoted by **(1, 2)*-s-cl(A)** (resp. **(1, 2)*-α-cl(A), (1, 2)*-η-cl(A)**), defined as the intersection of all (1, 2)*-semi-closed. (resp. (1, 2)*-α-closed, (1, 2)*-η-closed) sets containing A.

The family of all (1, 2)*-semi-open (resp. (1, 2)*-α-open, (1, 2)*-η-open, (1, 2)*-semi-closed, (1, 2)*-α-closed, (1, 2)*-η-closed) sets in X is denoted by (1, 2)*-SO(X) (resp. (1, 2)*-αO(X), (1, 2)*-ηO(X), (1, 2)*-SC(X), (1, 2)*-αC(X), (1, 2)*-ηC(X).

Remark 2.6. It is evident that any $\Im_{1,2}$-open set of X is an (1, 2)*-α-open and each (1, 2)*-α-open set of X is (1, 2)*-semi-open but the converses are not true.

Remark 2.7. We have the following implications for the properties of subsets [3]:

$\Im_{1,2}$-open ⇒ (1, 2)*-α-open ⇒ (1, 2)*-semi-open ⇒ (1, 2)*-η-open

Where none of the implications is reversible.

3. (1, 2)*-generalized η-closed Sets in Bitopological Spaces

Definition 3.1. A subset A of a bitopological space (X, $\Im_1$, $\Im_2$) is called
(i) (1, 2)*- generalized closed (briefly (1, 2)*-g-closed) [8] if $\Im_{1,2}$-cl(A) ⊂ U whenever A ⊂ U and U is $\Im_{1,2}$ -open in X.
(ii) (1, 2)*-weakly closed (briefly (1, 2)*-w-closed) [2] if $\Im_{1,2}$-cl(A) ⊂ U whenever A ⊂ U and U is (1, 2)*-semi-open in X.
(iii) (1, 2)*-α-generalized closed (briefly (1, 2)*-αg-closed) [8] if (1, 2)*-α-cl(A) ⊂ U whenever A ⊂ U and U is $\Im_{1,2}$-open in X.
(iv) (1, 2)*-generalized semi-closed (briefly (1, 2)*-gs-closed) [8] if (1, 2)*-s-cl(A) ⊂ U whenever A ⊂ U and U is $\Im_{1,2}$-open in X.
(v) (1, 2)*-generalized η-closed (briefly (1, 2)*-gη-closed) if (1, 2)*-η-cl(A) ⊂ U whenever A ⊂ U and U is $\Im_{1,2}$-open in X.

The complement of a $(1, 2)^*$-g-closed (resp. $(1, 2)^*$-w-closed, $(1, 2)^*$-αg-closed, $(1, 2)^*$-gs-closed, $(1, 2)^*$-gη-closed) set is called $(1, 2)^*$-g-open (resp. $(1, 2)^*$-w-open, $(1, 2)^*$-αg-open, $(1, 2)^*$-gs-open, $(1, 2)^*$-gη-open).
We denote the set of all $(1, 2)^*$-gη-closed sets in $(X, \Im_1, \Im_2)$ by $(1, 2)^*$-gη-C(X).

Theorem 3.2. Every $\Im_{1,2}$-closed set is gη-closed.
Proof. Let A be any $\Im_{1,2}$-closed set in $(X, \Im_1, \Im_2)$ and $A \subset U$, where U is $\Im_{1,2}$-open. So $(1, 2)^*$-cl(A) = A . Since every $\Im_{1,2}$-closed set is $(1, 2)^*$-η-closed, so $(1, 2)^*$-η-cl(A) $\subset$ $(1, 2)^*$-cl(A) = A. Therefore, $(1, 2)^*$-η-cl(A) $\subset A \subset$ U. Hence A is $(1, 2)^*$-gη-closed set.

Theorem 3.3. Every $(1, 2)^*$-α-closed set is $(1, 2)^*$-gη-closed.
Proof. Let A be any $(1, 2)^*$-α-closed set in $(X, \Im_1, \Im_2)$ and $A \subset U$, where U is $\Im_{1,2}$-open. Since every $(1, 2)^*$-α-closed set is $(1, 2)^*$-η-closed, so $(1, 2)^*$-η-cl(A) $\subset$ $(1, 2)^*$-α-cl(A) = A. Therefore $(1, 2)^*$-η-cl(A) $\subset A \subset U$. Hence A is $(1, 2)^*$-gη-closed set.

Theorem 3.4. Every $(1, 2)^*$-semi-closed set is $(1, 2)^*$-gη-closed.
Proof. Let A be any $(1, 2)^*$-semi-closed set in $(X, \Im_1, \Im_2)$ and $A \subset U$, where U is $\Im_{1,2}$-open. Since every $(1, 2)^*$-semi-closed set is $(1, 2)^*$-η-closed, so $(1, 2)^*$-η-cl(A) $\subset$ $(1, 2)^*$-s-cl(A) = A. Therefore $(1, 2)^*$-η-cl(A) $\subset A \subset U$. Hence A is $(1, 2)^*$-gη-closed set.

Theorem 3.5. Every $(1, 2)^*$-η-closed set is $(1, 2)^*$-gη-closed.
Proof. Let A be any $(1, 2)^*$-η-closed set in $(X, \Im_1, \Im_2)$ and $A \subset U$, where U is $\Im_{1,2}$-open. Since A is $(1, 2)^*$-η-closed. Therefore $(1, 2)^*$-η-cl(A) = $A \subset U$. Hence A is $(1, 2)^*$-gη-closed set.

Theorem 3.6. Every $(1, 2)^*$-g-closed set is $(1, 2)^*$-gη-closed.
Proof. Let A be any $(1, 2)^*$-g-closed set in $(X, \Im_1, \Im_2)$ then $(1, 2)^*$-cl(A) $\subset$ U whenever $A \subset U$, where U is $\Im_{1,2}$-open. So $(1, 2)^*$-η-cl(A) $\subset$ $(1, 2)^*$-cl(A) $\subset$ U. Therefore $(1, 2)^*$-η-cl(A) $\subset$ U. Hence A is $(1, 2)^*$-gη-closed set.

Theorem 3.7. Every $(1, 2)^*$-w-closed set is $(1, 2)^*$-gη-closed.
Proof. Let A be any $(1, 2)^*$-w-closed set in $(X, \Im_1, \Im_2)$ then $(1, 2)^*$-cl(A) $\subset$ U whenever $A \subset U$, where U is $\Im_{1,2}$-open, since every $\Im_{1,2}$-open set is $(1, 2)^*$-semi-open. So $(1, 2)^*$-η-cl(A) $\subset$ $(1, 2)^*$-cl(A) $\subset$ U. Therefore $(1, 2)^*$-η-cl(A) $\subset$ U. Hence A is $(1, 2)^*$-gη-closed set.

Theorem 3.8. Every $(1, 2)^*$-αg-closed set is $(1, 2)^*$-gη-closed.
Proof. Let A be any $(1, 2)^*$-αg-closed set in $(X, \Im_1, \Im_2)$ then $(1, 2)^*$-α-cl(A) $\subset$ U whenever $A \subset U$, where U is $\Im_{1,2}$-open. Given that A is $(1, 2)^*$-αg-closed set such that $(1, 2)^*$-α-cl(A) $\subset$ U. But we have $(1, 2)^*$-η-cl(A) $\subset$ $(1, 2)^*$-α-cl(A) $\subset$ U. Therefore $(1, 2)^*$-η-cl(A) $\subset$ U. Hence A is $(1, 2)^*$-gη-closed set.

Theorem 3.9. Every $(1, 2)^*$-gs-closed set is $(1, 2)^*$-gη-closed.
Proof. Let A be any $(1, 2)^*$-gs-closed set in $(X, \Im_1, \Im_2)$ then $(1, 2)^*$-s-cl(A) $\subset$ U whenever $A \subset U$, where U is $\Im_{1,2}$-open. Given that A is $(1, 2)^*$-gs-closed set such that $(1, 2)^*$-s-cl(A) $\subset$ U. But we have $(1, 2)^*$-η-cl(A) $\subset$ $(1, 2)^*$-s-cl(A) $\subset$ U. Therefore $(1, 2)^*$-η-cl(A) $\subset$ U. Hence A is $(1, 2)^*$-gη-closed set.

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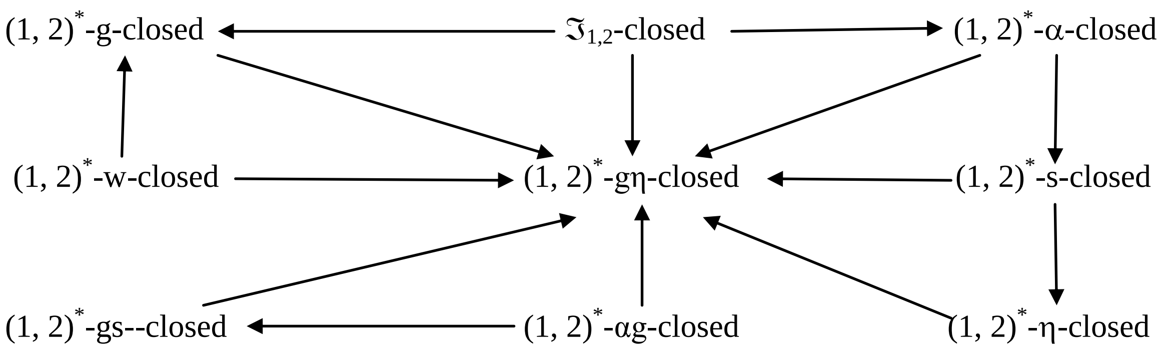
Remark 3.10. We have the following implications for the properties of subsets:



Where none of the implications is reversible as can be seen from the following examples:

Example 3.11. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}, {b}, {a, b}, {b, c, d}} and $\Im_2$ = {ϕ, X, {c}, {a, c, d}}. Then

(i) $\Im_{1,2}$-closed sets : ϕ, X, {a}, {b}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(ii) $(1, 2)^*$-α-closed sets : ϕ, X, {a}, {b}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(iii) $(1, 2)^*$-semi-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(iv) $(1, 2)^*$-η-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(v) $(1, 2)^*$-g-closed sets : ϕ, X, {a}, {b}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(vi) $(1, 2)^*$-w-closed sets : ϕ, X, {a}, {b}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(vii) $(1, 2)^*$-αg-closed sets : ϕ, X, {a}, {b}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(viii) $(1, 2)^*$-gs-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(ix) $(1, 2)^*$-gη-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

Example 3.12. Let X = {a, b, c} with $\Im_1$ = {ϕ, X, {b}} and $\Im_2$ = {ϕ, X, {c}}. Then

(i) $\Im_{1,2}$-closed sets : ϕ, X, {a}, {a, b}, {a, c}.
(ii) $(1, 2)^*$-α-closed sets : ϕ, X, {a}, {a, b}, {a, c}.
(iii) $(1, 2)^*$-semi-closed sets : ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}.
(iv) $(1, 2)^*$-η-closed sets : ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}.
(v) $(1, 2)^*$-g-closed sets : ϕ, X, {a}, {a, b}, {a, c}.
(vi) $(1, 2)^*$-w-closed sets : ϕ, X, {a}, {a, b}, {a, c}.
(vii) $(1, 2)^*$-αg-closed sets : ϕ, X, {a}, {a, b}, {a, c}.
(viii) $(1, 2)^*$-gs-closed sets : ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}.
(ix) $(1, 2)^*$-gη-closed sets : ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}.

Example 3.13. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}} and $\Im_2$ = {ϕ, X, {b}, {a, b, c}}. Then

(i) $\Im_{1,2}$-closed sets : ϕ, X, {d}, {c, d}, {a, c, d}, {b, c, d}.
(ii) $(1, 2)^*$-α-closed sets : ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}.
(iii) $(1, 2)^*$-semi-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(iv) $(1, 2)^*$-η-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(v) $(1, 2)^*$-g-closed sets : ϕ, X, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(vi) $(1, 2)^*$-w-closed sets : ϕ, X, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(vii) $(1, 2)^*$-αg-closed sets : ϕ, X, {c}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(viii) $(1, 2)^*$-gs-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(ix) $(1, 2)^*$-gη-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

Example 3.14. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {ϕ, X, {a, b, d}}. Then

(i) $\Im_{1,2}$-closed sets : ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}.
(ii) $(1, 2)^*$-α-closed sets : ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}.
(iii) $(1, 2)^*$-semi-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(iv) $(1, 2)^*$-η-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(v) $(1, 2)^*$-g-closed sets : ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}.
(vi) $(1, 2)^*$-w-closed sets : ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}.
(vii) $(1, 2)^*$-αg-closed sets : ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}.
(viii) $(1, 2)^*$-gs-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(ix) $(1, 2)^*$-gη-closed sets : ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

4. Some Properties of $(1, 2)^*$-generalized η-closed Sets

Theorem 4.1. The union of any two $(1, 2)^*$-gη-closed subsets of (X, $\Im_1$, $\Im_2$) is need not be $(1, 2)^*$-gη-closed subset of (X, $\Im_1$, $\Im_2$) as per the following example.

Example 4.2. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}} and $\Im_2$ = {ϕ, X, {b}, {a, b, c}}. Here A = {a} and B = {b} are $(1, 2)^*$-gη-closed subsets in (X, $\Im_1$, $\Im_2$). Then A ∪ B = {a, b} is not $(1, 2)^*$-gη-closed subsets in (X, $\Im_1$, $\Im_2$).

Remark 4.3. The intersection of two $(1, 2)^*$-gη-closed-sets in (X, $\Im_1$, $\Im_2$) is also a $(1, 2)^*$-gη-closed set in (X, $\Im_1$, $\Im_2$).

Proof. Easy to verify.

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Theorem 4.4. If a subset A is $(1, 2)^*$-gη-closed of X, then $(1, 2)^*$-η-cl(A) – A does not contain any non-empty $\Im_{1,2}$-closed set.
Proof. Let F be a $\Im_{1,2}$-closed subset of $(1, 2)^*$-η-cl(A) – A. Then $F \subset (1, 2)^*$-η-cl(A) and $F \cap S = \phi$. Therefore X – F is $\Im_{1,2}$-open and hence X – F is $\Im_{1,2}$-open. Since $F \cap A = \phi$, $A \subset X - F$. But A is $(1, 2)^*$-gη-closed, then $(1, 2)^*$-η-cl(A) ⊂ X – F and consequently F ⊂ X – $(1, 2)^*$-η-cl(A). Therefore F ⊂ ($(1, 2)^*$-η-cl(A)) ∩ (X – $(1, 2)^*$-η-cl(A)) and hence F is empty.

Remark 4.5. The converse of **Theorem 4.4** is not true as per the following example.

Example 4.6. Let X = {a, b, c, d, e} with $\Im_1$ = {ϕ, X, {a, b}, {a, b, c, d}} and $\Im_2$ = {ϕ, X, {c, d}, {a, b, c, d}}. If we consider A = {a, c}, then $(1, 2)^*$-η-cl(A) – A = X – {a, c} = {b, c} does not contain any non-empty $\Im_{1,2}$-closed set. However A is not $(1, 2)^*$-gη-closed.

Theorem 4.7. Let A be a $(1, 2)^*$-gη-closed subset of X. If A ⊂ B ⊂ $(1, 2)^*$-η-cl(A), then B is also $(1, 2)^*$-gη-closed in X.
Proof. Let U ∈ $(1, 2)^*$-gηO(X) with B ⊂ U. Then A ⊂ U. Since A is $(1, 2)^*$-gη-closed, $(1, 2)^*$-η-cl(A) ⊂ U. Also, since B ⊂ $(1, 2)^*$-η-cl(A), $(1, 2)^*$-η-cl(B) ⊂ $(1, 2)^*$-η-cl(A) ⊂ U. Hence B is also $(1, 2)^*$-gη-closed subset of X.

Theorem 4.8. For an element x ∈ (X, $\Im_1$, $\Im_2$), the set (X, $\Im_1$, $\Im_2$) – {x} is $(1, 2)^*$-gη-closed or $\Im_{1,2}$-open.
Proof. Suppose (X, $\Im_1$, $\Im_2$) – {x} is not $\Im_{1,2}$-open set. Then (X, $\Im_1$, $\Im_2$) is the only $\Im_{1,2}$-open set containing (X, $\Im_1$, $\Im_2$) – {x}. This implies $(1, 2)^*$-η-cl((X, $\Im_1$, $\Im_2$) – {x}) ⊂ (X, $\Im_1$, $\Im_2$). Hence (X, $\Im_1$, $\Im_2$) – {x} is $(1, 2)^*$-gη-closed set in (X, $\Im_1$, $\Im_2$).

Theorem 4.9. If A is an open and S is $(1, 2)^*$-η-open in bitopological space (X, $\Im_1$, $\Im_2$), then A ∩ S is $(1, 2)^*$-η-open in (X, $\Im_1$, $\Im_2$).

Theorem 4.10. If A is both open and $(1, 2)^*$-g-closed set in (X, $\Im_1$, $\Im_2$), then it is $(1, 2)^*$-gη-closed set in (X, $\Im_1$, $\Im_2$).
Proof. Let A be an open and $(1, 2)^*$-g-closed set in (X, $\Im_1$, $\Im_2$). Let A ⊂ U and let U be a $\Im_{1,2}$-open set in (X, $\Im_1$, $\Im_2$). Now A ⊂ A. By hypothesis $(1, 2)^*$-η-cl(A) ⊂ A. That is $(1, 2)^*$-η-cl(A) ⊂ U. Thus A is $(1, 2)^*$-gη-closed in (X, $\Im_1$, $\Im_2$).

5. $(1, 2)^*$-gη-open sets and $(1, 2)^*$-gη-neighborhood

In this section, we study $(1, 2)^*$-gη-open sets in bitopological spaces and obtain some of their properties. Also, we introduce $(1, 2)^*$-gη-neighborhood (shortly $(1, 2)^*$-gη-nbhd in bitopological spaces by using the notion of $(1, 2)^*$-gη-open sets. We prove that every nbhd of x in (X, $\Im_1$, $\Im_2$) is $(1, 2)^*$-gη-nbhd of x but not conversely.

Definition 5.1. A subset A in (X, $\Im_1$, $\Im_2$) is called $(1, 2)^*$-generalized η-open (briefly, $(1, 2)^*$-gη-open) in (X, $\Im_1$, $\Im_2$) if A^c is $(1, 2)^*$-gη-closed in (X, $\Im_1$, $\Im_2$). We denote the family of all $(1, 2)^*$-gη-open sets in X by $(1, 2)^*$-gηO(X).

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Definition 5.2. Let $(X, \Im_1, \Im_2)$ be a bitopological space and let $x \in (X, \Im_1, \Im_2)$. A subset N of $(X, \Im_1, \Im_2)$ is said to be a **$(1, 2)^*$-gη-nbhd** of x iff there exists a $(1, 2)^*$-gη-open set G such that $x \in G \subset N$.

Definition 5.3. A subset N of a bitopological space $(X, \Im_1, \Im_2)$, is called a **$(1, 2)^*$-gη-nbhd** of $A \subset (X, \Im_1, \Im_2)$ iff there exists a (1, 2)*-gη-open set G such that $A \subset G \subset N$.

Remark 5.4. The $(1, 2)^*$-η-nbhd N of $x \in (X, \Im_1, \Im_2)$ need not be a $(1, 2)^*$-η-open in $(X, \Im_1, \Im_2)$.

Example 5.5. Let X = {a, b, c, d, e} and $\Im_1$= {ϕ, {a, b}, {a, b, c, d}, X} and $\Im_2$ = {ϕ, {c, d}, {a, c, d}, X}. Then $(1, 2)^*$-gηO(X) = {ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, b, e}, {a, c, d}, {b, c, d}, {c, d, e}, {a, b, c, d}, {a, b, c, e}, {a, b, d, e}, {a, c, d, e}, {b, c, d, e}}. Note that {a, e} is not a $(1, 2)^*$-gη-open set, but it is a $(1, 2)^*$-η-nbhd of a, since {a} is a $(1, 2)^*$-gη-open set such that a ∈ {a} ⊂ {a, e}.

Theorem 5.6. Every nbhd N of $x \in (X, \Im_1, \Im_2)$ is a $(1, 2)^*$-gη-nbhd of $(X, \Im_1, \Im_2)$.
Proof. Let N be a nbhd of point $x \in (X, \Im_1, \Im_2)$. To prove that N is a $(1, 2)^*$-gη-nbhd of x. By definition of nbhd, there exists an open set G such that $x \in G \subset N$. As every open set is $(1, 2)^*$-gη-open set G such that $x \in G \subset N$. Hence N is $(1, 2)^*$-gη-nbhd of x.

Remark 5.7. In general, a $(1, 2)^*$-gη-nbhd N of $x \in (X, \Im_1, \Im_2)$ need not be a nbhd of x in $(X, \Im_1, \Im_2)$, as seen from the following example.

Example 5.8. Let X = {a, b, c, d} with topology $\Im_1$ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X} and $\Im_2$ = {ϕ, {a, b, d}, X} Then $(1, 2)^*$-gηO(X) = {X, ϕ, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. The set {b, c} is $(1, 2)^*$-gη-nbhd of the point b, there exists an $(1, 2)^*$-gη-open set {b} is such that b ∈ {b} ⊂ {b, c}. However, the set {b, c} is not a nbhd of the point b, since no open set G exists such that $b \in G \subset$ {a, c}.

Theorem 5.9. If a subset N of a space $(X, \Im_1, \Im_2)$ is $(1, 2)^*$-gη-open, then N is a $(1, 2)^*$-gη-nbhd of each of its points.
Proof. Suppose N is $(1, 2)^*$-gη-open. Let x ∈ N. We claim that N is $(1, 2)^*$-gη-nbhd of x. For N is a $(1, 2)^*$-gη-open set such that $x \in N \subset N$. Since x is an arbitrary point of N, it follows that N is a $(1, 2)^*$-gη-nbhd of each of its points.

Remark 5.10. The converse of the above theorem is not true in general as seen from the following example.

Example 5.11. Let X = {a, b, c, d, e} with topology $\Im_1$ = {ϕ, {a, b}, {a, b, c, d}, X} and $\Im_2$ = {ϕ, {c, d}, {a, b, d}, X}. Then $(1, 2)^*$-gηO(X) = {ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, b, e}, {a, c, d}, {b, c, d}, {c, d, e}, {a, b, c, d}, {a, b, c, e}, {a, b, d, e}, {a, c, d, e}, {b, c, d, e}}. The set {a, b, c} is a $(1, 2)^*$-gη-nbhd of the points a, b and c since the $(1, 2)^*$-gη-open sets {a}, {b} and {c} for the points a, b and c respectively, such that a ∈ {a} ⊂ {a, b, c}; b ∈ {b} ⊂ {a, b, c} and c ∈ {c} ⊂ {a, b, c} respectively. That is {a, b, c} is a $(1, 2)^*$-gη-nbhd of each of its points. However the set {a, b, c} is not a $(1, 2)^*$-gη-open set in X.

Definition 5.12. Let x be a point in a space $(X, \Im_1, \Im_2)$. The set of all $(1, 2)^*$-gη-nbhd of x is called the **$(1, 2)^*$-gη-nbhd system** at x, and is denoted by $(1, 2)^*$-gη-N(x).

Theorem 5.13. Let $(X, \Im_1, \Im_2)$ be a bitopological space and for each $x \in (X, \Im_1, \Im_2)$. Let $(1, 2)^*$- gη-N(x) be the collection of all $(1, 2)^*$-gη-nbhds of x. Then we have the following results.
(i) $\vee\ x \in (X, \Im_1, \Im_2)$, $(1, 2)^*$-gη-N(x) $\neq \phi$.
(ii) $N \in (1, 2)^*$-gη-N(x) $\Rightarrow x \in N$.
(iii) $N \in (1, 2)^*$-gη-N(x), $M \supset N \Rightarrow M \in (1, 2)^*$-gη-N(x).
(iv) $N \in (1, 2)^*$-gη-N(x), $M \in (1, 2)^*$-gη-N(x) $\Rightarrow N \cap M \in (1, 2)^*$-gη-N(x).
(v) $N \in (1, 2)^*$-gη-N(x) $\Rightarrow$ there exists $M \in (1, 2)^*$-gη-N(x) such that $M \subset N$ and $M \in (1, 2)^*$-gη-N(y) for every $y \in M$.

Proof. (i) Since $(X, \Im_1, \Im_2)$ is a $(1, 2)^*$-gη-open set, it is a $(1, 2)^*$-gη-nbhd of every $x \in (X, \Im_1, \Im_2)$. Hence there exists at least one $(1, 2)^*$-gη-nbhd (namely - $(X, \Im_1, \Im_2)$) for each $x \in (X, \Im_1, \Im_2)$. Hence$(1, 2)^*$-gη-N(x) = ϕ for every $x \in (X, \Im_1, \Im_2)$.

(ii) If $N \in (1, 2)^*$-gη-N(x), then N is a (1, 2)*-gη-nbhd of x. So by definition of $(1, 2)^*$-gη-nbhd, $x \in N$.

(iii) Let $N \in (1, 2)^*$-gη-N(x) and $M \supset N$. Then there is a $(1, 2)^*$-gη-open set G such that $x \in G \subset N$. Since $N \subset M$, $x \in G \subset M$ and so M is $(1, 2)^*$-gη-nbhd of x. Hence $M \in (1, 2)^*$-gη-N (x).

(iv) Let $N \in (1, 2)^*$-gη-N(x) and $M \in$ (1, 2)*-gη-N(x). Then by definition of $(1, 2)^*$-gη-nbhd. Hence $x \in G_1 \cap G_2 \subset N \cap M \Rightarrow (1)$. Since $G_1 \cap G_2$ is a $(1, 2)^*$-gη-open set, (being the intersection of two (1, 2)*-gη-open sets), it follows from (1) that $N \cap M$ is a $(1, 2)^*$-gη-nbhd of x. Hence $N \cap M \in (1, 2)^*$-gη-N(x).

(v) If $N \in (1, 2)^*$-gη-N(x), then there exists a $(1, 2)^*$-gη-open set M such that $x \in M \subset N$. Since M is a $(1, 2)^*$-gη-open set, it is $(1, 2)^*$-gη-nbhd of each of its points. Therefore $M\in (1, 2)^*$-gη-N(y) for every $y\in M$.

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