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CONNECTED COMPLEMENTARY TREE DOMINATION NUMBER OF GRAPHS

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ABSTRACT

Let G = (V, E) be a non-trivial, simple, finite and undirected graph. A dominating set D is called a complementary tree dominating set if the induced subgraph <V–D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. A dominating set D is called a connected complementary tree dominating set (cctd-set) if the induced subgraph <D> is connected. The connected complementary tree dominating number $\gamma_{cctd}(G)$ of a connected graph G is the minimum cardinality of a connected complementary tree dominating set of G.

In this paper, connected complementary tree dominating set, connected complementary number are defined and minimal connected complementary tree dominating set are characterized and bounds also obtained.

KEYWORDS: *Connected domination, connected complementary tree domination*.

AMS Subject Classification (2010): 05C69.

1. INTRODUCTION

The graph considered here are nontrivial, simple, finite and undirected. Let G be a graph with vertex set V(G) and edge set E(G). For v ∈ V(G) the neighbourhood N(v) of v is the set of all vertices adjacent to v in G. N[v] = N(v) ∩ {v} is called the closed neighborhood of v. The concept of domination was first studied by Ore [6]. A set D ⊆ V is said to a dominating set of G. If every vertex in V–D is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. The concept of complementary tree domination was introduced by S. Muthammai, M. Bhanumathi and P. Vidhya in [4]. A dominating set D ⊆ V is called a complementary tree dominating (ctd) set, if the induced subgraph <V–D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. The concept of connected domination in graphs was introdued by E. Sampathkumar and Walikar [5]. A dominating set D of a connected graph G = (V, E) is a connected dominating set, if the induced subgraph <D> is connected. The connected domination number $\gamma_c(G)$ of a connected graph G is the minimum cardinality of a connected dominating set. A dominating set D is called a connected complementary tree dominating set if the induced subgraph <D> is connected. The connected complementary tree domination number $\gamma_{cctd}(G)$ of a connected graph G is the minimum cardinality of a connected complementary tree dominating set. In this paper, connected complementary tree dominating set, connected complementary number are defined and minimum connected complementary tree dominating set and its bounds are obtained.

2. RESULTS

Theorem 2.1. [3]

If G is a connected graph with $p \geq 3$ vertices, then $\gamma_c(G) = p - \varepsilon_T(G)$ where $\varepsilon_T(G)$ is the maximum number of pendant edges in any spanning tree of G.

Theorem 2.2. [5]

For any connected graph G, $\frac{p}{\Delta(G)+1} \leq \gamma_c(G) \leq 2q - p$.

Theorem 2.3. [2]

For any connected graph G, $\gamma_c(G) \leq p - \Delta(G)$.

3. CONNECTED COMPLEMENTARY TREE DOMINATION NUMBER OF GRAPHS

In this section, connected complementary tree dominating set, connected complementary tree domination number are defined and minimal connected complementary tree dominating sets are characterized. Also bounds of connected complementary tree domination number are obtained.

In the following, connected complementary tree domination number is defined.

Definition 3.1:-

A complementary tree dominating set $D \subseteq V$ of a connected graph G = (V, E) is said to be a connected complementary tree dominating set (cctd-set), if the induced subgraph <D> is connected.

The connected complementary tree domination number $\gamma_{cctd}(G)$ of a connected graph G is the minimum cardinality of a connected complementary tree dominating set of G.

A γ_{cctd}-set is a minimum connected complementary tree dominating set.

A connected complementary tree dominating set is said to be minimal, if no proper subset of D is connected complementary tree dominating set.

It is to be noted that γ_{cctd}-set exists, for all connected graphs.

Example 3.1.

Consider the graph in Figure 1.

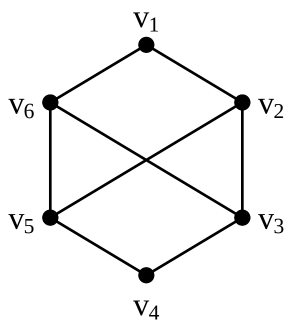

Figure 1

The sets $D_1 = \{v_2, v_5\}$ and $D_2 = \{v_3, v_6\}$, $D_3 = \{v_2, v_3\}$, $D_4 = \{v_5, v_6\}$ are connected complementary tree dominating sets of minimum cardinality.

Therefore, $\gamma_{cctd}(G) = 2$.

Observation 3.1.

(i) Let G be a connected graph with $\Delta(G) < p-1$.
Since, every cctd-set is a complementary tree dominating set,
$\gamma_{ctd}(G) \leq \gamma_{cctd}(G)$ for any connected graph G.
Also, every cctd-set is a connected dominating set.
Therefore $\gamma_c(G) \leq \gamma_{cctd}(G)$ for any connected graph G.
Equality holds, if G is the complete bipartite graph $K_{2,n}$ ($n \geq 2$).

(ii) Let H be a connected spanning subgraph of a connected graph G. It is not necessary that the inequality $\gamma_{cctd}(G) \leq \gamma_{cctd}(H)$ holds. For example, $\gamma_{cctd}(K_5) = 3$ and $\gamma_{cctd}\}(K_5 - e) = 2$.

(iii) If H is a connected induced subgraph of G, then also the inequality $\gamma_{cctd}(G) \leq \gamma_{cctd}(H)$ does not hold always. For example, $\gamma_{cctd}(K_5) = 3$ and $\gamma_{cctd}(K_4) = 2$.

(iv) For any connected graph G with atleast two vertices, $1 \leq \gamma_{cctd}(G) \leq p-1$.

(v) If $\gamma_{cctd}(G) < p - 1$, a connected ctd-set of G contains pendant vertices and its supports.

The following theorem characterizes the minimal connected ctd-sets and is stated without proof.

Theorem 3.1.

A connected ctd-set $D \subseteq V(G)$ is minimal if and only if for each vertex v in D, one of the following conditions holds

(i) There exists a vertex u in V – D such that $N(u) \cap D = \{v\}$.

(ii) $N(v) \cap (V - D) = \phi$

(iii) The subgraph $<(V - D) \cup \{v\}>$ or G induced by $(V - D) \cup \{v\}$ either contains a cycle or disconnected.

(iv) The subgraph $<D - \{v\}>$ of G induced by $D - \{v\}$ is disconnected.

Observation 3.2.

(i) For any path P_n, $\gamma_{cctd}(G) = n - 1$, $n \geq 4$.

(ii) For any cycle C_n, $\gamma_{cctd}(G) = n - 2$, $n \geq 4$.

(iii) For any complete graph K_n, $\gamma_{cctd}(G) = n - 2$, $n \geq 4$.

(iv) For any star $K_{1,n}$, $\gamma_{cctd}(K_{1,n}) = n$, $n \geq 2$.

(v) For any complete bipartite graph $K_{m,n}$, $\gamma_{cctd}(K_{m,n}) = \min(m, n)$, $(m, n \geq 2)$.

(vi) $\gamma_{cctd}(C_n \circ K_1) = 2n - 1$, $n \geq 3$.

(vii) For any wheel W_n, $\gamma_{cctd}(W_n) = 2$, $n \geq 4$.

(viii) Let G be the subdivision graph of star $K_{1,n}$. Then $\gamma_{cctd}(G) = 2n$, $n \geq 2$.

(ix) $\gamma_{cctd}(C_n^{(t)}) = (n - 1)t - 1$.

Here, $C_n^{(t)}$ has $p = (n - 1)t + 1$ vertices. Here, the point of union of cycles and all the vertices of the cycles except two vertices from any one of the cycle forms a cctd-set of G. Therefore, $\gamma_{cctd}(C_n^{(t)}) = p - 2 = (n - 1)t - 1$.

Theorem 3.2.

Let G_1 and G_2 be any two connected graphs of order atleast 3. Then, $\gamma_{cctd}(G_1 \circ G_2) \leq |V(G_1)| (1 + |V(G_2)|) - |V(T)|$, where T is a subgraph of G_2 which is a tree with maximum number of vertices.

Proof.

The set $V(G_1 \circ G_2) - V(T)$ is a cctd-set of $G_1 \circ G_2$ and hence

$$\gamma_{cctd}(G) \leq |V(G_1 \circ G_2| - |V(T)|$$
$$= |V(G_1)|(1 + |V(G_2)|) - |V(T)|$$

Equality holds, if $G_1 \cong C_3$ and $G_2 \cong C_3$. □

Corollary 3.1.

If G_2 is a tree, then

$$\gamma_{cctd}(G_1 \circ G_2) = |V(G_1)|(1 + |V(G_2)|) - (|V(G_2)| - 1)$$

4. BOUNDS AND EXACT VALUES FOR THE CONNECTED COMPLEMENTARY TREE DOMINATION NUMBER

Observation 4.1.

For any connected graph G, $\gamma_{cctd}(G) \geq \left\lceil \frac{p}{\Delta(G)+1} \right\rceil$.

Since $\gamma_{cctd}(G) \geq \gamma_c(G)$ and $\gamma_c(G) \geq \left\lceil \frac{p}{\Delta(G)+1} \right\rceil$,

the inequality $\gamma_{cctd}(G) \geq \left\lceil \frac{p}{\Delta(G)+1} \right\rceil$ holds.

This bound is attained, if $G \cong C_4, C_5$.

Theorem 4.1.

For any connected (p, q) graph G, $\gamma_{cctd}(G) \geq 2p - q - 2$.

Proof.

Let D be a γ_{cctd}-set of G since <D> is connected, number of edges in <D> is greater than or equal to $|D| - 1 = \gamma_{cctd}(G) - 1$.

Number of edges in <V–D> is $p - \gamma_{cctd}(G) - 1$.

There are atleast $p-\gamma_{cctd}(G)$ edges from $V - D$ to D.

Therefore, $q \geq \gamma_{cctd}(G) - 1 + p - \gamma_{cctd}(G) + p - \gamma_{cctd}(G) - 1$

That is, $q \geq 2p - \gamma_{cctd}(G) - 2$.

Hence, $\gamma_{cctd}(G) \geq 2p - q - 2$.

Equality holds, if $G \cong K_{1,n}$; $n \geq 3$. □

Theorem 4.2.

Let G be a connected graph with atleast three vertices and let T be a spanning tree of G with maximum number $\varepsilon_T(G)$ of pendant vertices. Then, $\gamma_{cctd}(G) = p - \varepsilon_T(G)$, if and only if the subgraph of G induced by pendant vertices of T is a tree in G.

Proof.

Let S be the set of all pendant vertices in T. Therefore, $|S| = \varepsilon_T(G)$. Then $V - S$ is a connected dominating set.

Assume <S> is a tree. Then, $V - S$ is a cctd-set of G and hence,

$\gamma_{cctd}(G) \leq |V - S| = p - \varepsilon_T(G)$. But, $\gamma_{cctd}(G) \geq \gamma_c(G) = p - \varepsilon_T(G)$.

Therefore, $\gamma_{cctd}(G) = p - \varepsilon_T(G)$.

Conversely, assume $\gamma_{cctd}(G) = p - \varepsilon_T(G)$. Then, there exists a cctd-set D such that $|D| = p - \varepsilon_T(G)$.

If the subgraph <S> of G is not a tree, then $V - S$ will not be a ctd-set.

Hence, <S> is a tree in G. □

Theorem 4.3.

Let G be a connected graph such that diam(G) = 2. If the subgraph of G induced by neighbourhood set of a vertex of maximum degree is a tree in G, then

$\gamma_{cctd}(G) \leq p - \Delta(G) + 1$.

Proof.

Let v be a vertex of maximum degree in G such that <N(v)> is a tree. Let u be a pendant vertex in <N(v)>, then $(V - N(v)) \cup \{u\}$ is a cctd-set of G. Hence,

$\gamma_{cctd}(G) \leq |(V - N(v)) \cup \{v\}|$ and hence, $\gamma_{cctd}(G) \leq p - \Delta(G) + 1$. □

Theorem 4.4.

Let G be a connected graph which is not complete such that $\gamma(G) = 1$ and $\delta(G) \geq 2$. Then,

$\gamma_{cctd}(G) \leq p - 3$.

Proof.

Since G is not complete, there exists atleast one pair of nonadjacent vertices. Hence, there exists an induced path P_3 on three vertices.

Then, $V(G) - V(P_3)$ is a cctd-set of G and hence $\gamma_{cctd}(G) \leq p - 3$.

This bound is attained, if $G \cong K_n - e$, $n \geq 4$. □

Theorem 4.5.

Let G be any nontrivial connected graph of order atleast three.

Then, $\gamma_{cctd}(G) = 1$ if and only if $G \cong T + K_1$, where T is a tree.

Proof.

The proof is in similar lines to Proposition 3.6 [4]. □

Next, the graphs G for which $\gamma_{cctd}(G) = 2$ are found.

Theorem 4.6.

Let G be a connected graph with atleast 4 vertices. Then, $\gamma_{cctd}(G) = 2$ if and only if G is one of the following graphs

(i) G is the graph $K_1 + T$ with one pendant edge attached at the vertex of K_1, where T is a tree.

(ii) G is the graph obtained from a tree by joining each of the vertices of the tree to the vertices of K_2 such that $\deg_G v \geq 2$, for all $v \in V(K_2)$.

Proof.

Let G be one of the graph mentioned in (i) and (ii). Since G is not isomorphic to $K_1 + T$. For any tree T, $\gamma_{cctd}(G) \geq 2$.

If G is the graph as in (i), the subset of V(G) consisting of the vertex of K_1 and the pendant vertex of G forms a cctd-set of G.

Therefore, $\gamma_{cctd}(G) \leq 2$ and hence $\gamma_{cctd}(G) = 2$. Conversely, assume $\gamma_{cctd}(G) = 2$ then, there exists a cctd-set D such that $|D| = 2$.

Let $D = \{u, v\}$.

a) If u or v is a pendant vertex in G, then all the vertices of V−D are adjacent to v or u. Therefore, G is the graph mentioned in (i).

b) Let $deg_G(u) \geq 2$ and $deg_G(v) \geq 2$. Since <V−D> is a tree and D is a tree and D is a connected dominating set of G, each vertex in V−D is adjacent to atleast one vertex in D. Hence, G is the graph as in (ii).

□

Theorem 4.7.

Let G be a connected graph. Then, $\gamma_{cctd}(G) = p-1$ if and only if either

(i) the subgraph of G induced by vertices of G which are not the cutvertices is totally disconnected (or) contains one vertex of G or

(ii) each vertex of G of degree atleast 2 is a cutvertex.

Proof.

Let either the subgraph of G induced by vertices which are not the cutvertices of G be totally disconnected (or) each vertex in G of degree atleast 2 is a cutvertex. Then, $\delta(G) = 1$. Let D be a cctd-set of G. Since the set $F \subseteq V(G)$ of cutvertices forms a connected dominating set, $F \subseteq D$ and also $<V - F>$ is a totally disconnected.

Since $<V - D>$ is a tree, all the vertices of V − F except one vertex must belong to D and hence, $|D| \geq p - 1$. But, $|D| \leq p - 1$ and hence, $|D| = p - 1$.

Conversely, assume $\gamma_{cctd}(G) = p - 1$.

Then, there exists a cctd-set D of G such that $|D| = p - 1$.

Let $<V - D> = \{v\}$. Since D is a dominating set, v is adjacent to atleast one vertex in D. If the subgraph H of G induced by vertices which are not the cutvertices is not totally disconnected (or) contains atleast two vertices, then there exists atleast one edge (u, v) in H. Then, the set $V - \{u, v\}$ will be a cctd-set of G and hence, $\gamma_{cctd}(G) \leq p - 2$, which is a contradiction. Hence, the theorem is proved. □

Corollary 4.1.

If G is a tree with p vertices, then $\gamma_{cctd}(G) = p - 1$.

Theorem 4.8.

Let $p \geq 4$ be an integer. For each k satisfying $2 \leq k \leq p - 2$, there is a connected graph G such that $\gamma_{cctd}(G) = \gamma_{ctd}(G) = k$.

Proof.

Construct a graph G as follows. Attach k − 2 pendant edges at exactly one vertex of K_4 or C_3 $(2 \leq k \leq p - 2)$.

For this graph, $\gamma_{cctd}(G) = \gamma_{ctd}(G) = k$. □

Theorem 4.9.

Let G be a connected graph with atleast three vertices. Then, $\gamma_{cctd}(G) = p - 2$ if and only if

(i) $G \cong K_p$ (or) C_p

(ii) If S is the set of all cutvertices of G such that $<V - S>$ is not totally disconnected, then atleast one of the following holds.

  (a) components of $<V - S>$ are complete graphs K_n, $n \geq 1$, $n \neq 2$.

  (b) If there exists an induced path P of length 2 in a component of $<V - S>$, then the central vertex of P is of degree 2 in G.

Proof.

Assume $\gamma_{cctd}(G) = p - 2$. Then, there exists a γ_{cctd}-set D such that $|D| = \gamma_{cctd}(G)$ and $<V - D> \cong K_2$.

Let S be the set of all cutvertices of G such that $<V - S>$ is not totally disconnected.

Case 1. $S = \phi$.

If there exists an induced path P of length 2 such that the central vertex of P is of degree atleast 2, then V(G) − V(P) is a cctd-set of G and hence $\gamma_{cctd}(G) \leq p - 3$.

Therefore, one of the following holds.

(i) There exists no induced path of length 2 in G.

(ii) central vertex of every induced path is of degree 2 in G.

If (i) holds, then $G \cong K_p$.

If (ii) holds, then $G \cong C_p$.

Case 2. $S \neq \phi$.

Then $S \subseteq D$.

By the same argument above, if there exists an induced path P of length 2 in a component of $<V - S>$ and if the central vertex has degree atleast 3 in G, then

$\gamma_{cctd}(G) \leq p - 3$. Hence, atleast one of (a) and (b) of (ii) holds.

Conversely, if $G \cong C_p$ (or) K_p, then $\gamma_{cctd}(G) = p - 2$.

If atleast one of (a) and (b) of (ii) holds, then there exists a minimum cctd-set of G consisting of (p − 2) vertices and hence, $\gamma_{cctd}(G) = p - 2$. □

Theorem 4.10.

Let G be a graph such that G and its complement $\overline{G}$ are connected. Then

$$4 \leq \gamma_{cctd}(G) + \gamma_{cctd}(\overline{G}) \leq 2(p-1)$$

$$4 \leq \gamma_{cctd}(G) + \gamma_{cctd}(\overline{G}) \leq (p-1)^2$$

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