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TOTAL AND CONNECTED COMPLEMENTARY TREE DOMINATION NUMBER OF UNICYLIC GRAPHS

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ABSTRACT

In this paper, some connected unicyclic graphs for which $\gamma_{tctd}(G) = \gamma_{ctd}(G)$ are found. Further, the connected unicyclic graphs for which $\gamma_{cctd}(G) = \gamma_{ctd}(G)$ and $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$ are characterized.

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1 INTRODUCTION

Graphs discussed in this paper are undirected and simple. For a graph G(V, E), let V and E denotes its vertex set and edge set respectively. A graph G is unicyclic if it contains exactly one cycle.

L. Volkman has studied graphs having equal domination number and edge independence number [??]. He has also investigated graphs with equal domination number and covering number. J. Paulraj Joseph and S. Arumugam have investigated graphs with equal domination and connected domination numbers [??].

In this paper, connected unicyclic graphs for which $\gamma_{ctd}(G) = \gamma(G)$ and $\gamma_{ctd}(G) = \gamma(G) + 1$ are established.

In this paper, some connected unicyclic graphs for which $\gamma_{tctd}(G) = \gamma_{ctd}(G)$ are found. Further, the connected unicyclic graphs for which $\gamma_{cctd}(G) = \gamma_{ctd}(G)$ and $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$ are characterized.

Notation 1.1.

Let P_m be a path on m (m ≥ 2) vertices and let $P_1 = K_1$ and $P_m^+ = P_m \circ K_1$ (m ≥ 1) be the Corona of P_m and K_1.

(i) By joining P_m^+ (m ≥ 1) at a vertex v of C_n, (n ≥ 3), it is meant that, joining a vertex of degree 2 of P_m^+ to v with an edge.

(ii) By joining $K_{1,n}$ (n ≥ 1) at a vertex v of C_n, it is meant that, joining the central vertex of $K_{1,n}$ to v with an edge.

(iii) By attaching a pendant edge (or a path P_n, n ≥ 3) at a vertex v of a graph G, it is meant that, merging a vertex of the pendant edge (or a pendant vertex of P_n, n ≥ 3) with v.

(iv) By attaching a tree to a vertex v of a graph G, it is meant that, merging a pendant vertex of the tree with v.

Notation 1.2.

The following classes of unicyclic graphs can be defined.

Let $H_1^{(t)}$ be the graph obtained from C_n (n ≥ 5) by attaching a pendant edge at each of the t vertices of C_n such that (n−t) consecutive vertices of C_n have degree 2 (t ≤ n).

(i) Let $\mathcal{G}_1^{(t)}$ be the class of unicyclic graphs $H_1^{(t)}$.

(ii) Let $\mathcal{G}_2^{(t)}$ be the class of unicyclic graphs obtained from $H_1^{(t)}$ by joining atleast one P_m^+ (m ≥ 1) at atleast one vertex of t consecutive vertices (t ≤ n) mentioned above.

(iii) Let $\mathcal{G}_3^{(t)}$ be the class of unicyclic graphs obtained from $H_1^{(t)}$ by joining atleast one P_m^+ (m ≥ 1) at atleast one of the two end vertices of above t consecutive vertices of C_n.

Theorem 1.1.

Let G be a connected unicyclic graph with the cycle C_n, n ≥ 5. Let T be the collection of trees T in G such that each vertex of T is adjacent to a vertex of degree atleast 2 in G and <V(G) − V(T)> has no isolated vertices. If t = max{ |T| : T ∈ T}, then $\gamma_{tctd}(G) = p - t$.

Proof.

Let T be the collection of trees T in G such that each vertex of T is adjacent to a vertex of degree atleast 2 in G such that |T| is maximum.

Let D = V(G) − V(T) and V(G) − D = V(T) and each vertex in V(G) − D is adjacent to a vertex in D and since <V(G) − D> is a tree, D is a ctd-set of G. Also <V(G) − V(T)> ≅ <D> contains no isolated vertices and hence, D is a total ctd-set of G.

Therefore, $\gamma_{tctd}(G) \leq |D| = |V(G) - V(T)| = p-t$. Also $\gamma_{tctd}(G) \geq p - t$ and hence, $\gamma_{tctd}(G) = p - t$. □

In the following, $\gamma_{tctd}(G) = \gamma_{ctd}(G)$ are found for the connected unicyclic graphs.

1. Let G be a connected unicyclic graph with C_3 as the unique cycle. If G is one of the following graphs, then $\gamma_{tctd}(G) = \gamma_{ctd}(G)$.
   G is a graph obtained from C_3 by
   (i) attaching pendant edges at a vertex of C_3
   (ii) attaching paths of length atleast 3 at a vertex of C_3 and pendant edges at another vertex of C_3
   (iii) attaching paths of length atleast 3 at all the vertices of C_3
   (iv) attaching pendant edges at a vertex of C_3 and attaching paths of length atleast 3 at another vertex such that vertices of the paths at distance atleast two from the vertex of C_3 is a support.
2. Let G be a connected unicyclic graph with C_4 as the unique cycle. If G is one of the following graphs, then $\gamma_{tctd}(G) = \gamma_{ctd}(G)$.
   G is a graph obtained from C_4 by
   (i) attaching paths of length atleast 3 at n vertices of C_4, where n = 1, 2, 4.
   (ii) attaching paths of length atleast 3 at two adjacent vertices of C_4 and pendant edges at atleast one of the remaining vertices of C_4.
   (iii) attaching a path of length atleast one at a vertex v of C_4 and then paths of length atleast 2 at the vertex at distance two from v.
   (iv) attaching pendant edges at a vertex v of C_4 and attaching paths of length atleast 3 at the vertex w at distance two from v such that the vertices of the paths at distance atleast two from w are supports.
   (v) $G \cong C_4$
3. Let G be a connected unicyclic graph with the cycle C_5. If G is one of the following graphs, then $\gamma_{tctd}(G) = \gamma_{ctd}(G)$.
   G is obtained from C_5 by
   (i) attaching a path of length 2 either at a vertex of C_5 (or) each at two vertices u, v of C_5 such that $d_{C_5}(u, v) = 2$
   (ii) attaching paths of length atleast 3 at n vertices of C_5, n = 1, 2, 3, 5.
   (iii) joining stars either at a vertex of C_5 or at two vertices u, v of C_5 such that $d_{C_5}(u, v) = 2$.
4. Let G be a connected unicyclic graph with the cycle C_n, n ≥ 6.
   If G is one of the following graphs, then $\gamma_{tctd}(G) = \gamma_{ctd}(G)$.
   (i) attaching paths of length atleast 3 at k vertices of C_n, where k = 1, 2, ..., n − 2, n
   (ii) joining stars at k vertices of C_n, where 1 ≤ k ≤ n − 4.
   (iii) attaching a path of length 2 at atleast k adjacent vertices of C_n, where 1 ≤ k ≤ n − 4.

Theorem 1.2.

Given an integer a ≥ 1, there exists a connected graph G such that $\gamma_{tctd}(G) = \gamma_{ctd}(G) + a$.

Proof.

Let G be the graph obtained by attaching (a+1) paths of length 2 at a vertex of C_n, n ≥ 5. Then, the set having (n−2) adjacent vertices of C_n and (a+1) pendant vertices form a minimum ctd-set and hence

$$\gamma_{ctd}(G) = n - 2 + a + 1$$
$$= n + a - 1$$

The set having (n−3) adjacent vertices of C_n, (a+1) pendant vertices and (a+1) supports form a minimum tctd-set of G and hence

$$\gamma_{tctd}(G) = n - 3 + 2(a + 1)$$
$$= n + 2a - 1$$

Therefore, $\gamma_{tctd}(G) = \gamma_{ctd}(G) + a$, a ≥1. □

Theorem 1.3.

Let G be a connected unicyclic graph having p vertices with the cycle C_n, n ≥ 3. Then, $\gamma_{cctd}(G) \geq p - 2$, p ≥ 4.

Proof.

Let G be a connected unicyclic graph with the cycle C_n, n ≥ 3 having p vertices.
Let D be a connected ctd-set of G. Then, |D| ≤ p − 1 and <V − D> is a tree. Since D is a ctd-set, D contains all the pendant vertices of G. Also, since <D> is connected, D contains all the cut vertices of G. Therefore, V − D contains vertices of C_n, having degree 2 in G.
If <V − D> has P_3 as a subgraph, then the central vertex of P_3 is not adjacent to any vertex in D.
Hence, <V − D> ≅ K_2 or K_1 and |D| ≥ p − 2.
Therefore, $\gamma_{cctd}(G) \geq p - 2$. □

In the following, connected unicyclic graphs for which $\gamma_{cctd}(G) = p - 2$ and p − 1 are characterized.

Theorem 1.4.

Let G be a connected unicyclic graph having p (p ≥ 4) vertices with the cycle C_n, n ≥ 3. Then, $\gamma_{cctd}(G) = p - 2$ if and only if there exists atleast two adjacent vertices of C_n which are of degree 2 in G.

Proof.

Assume there exists two adjacent vertices say u, v of C_n, having degree 2 in G. Let D = V − {u, v} is a cctd-set of G. Therefore, |D| ≤ p − 2 and hence, $\gamma_{cctd}(G) \leq p - 2$. □

Using Theorem ??, $\gamma_{cctd}(G) = p - 2$.

Theorem 1.5.

Let G be a connected unicyclic graph having p (p ≥ 4) vertices with the cycle C_n, n ≥ 3. Then, $\gamma_{cctd}(G) = p - 1$ if and only if either

(i) the vertices of C_n having degree 2 in G are independent in G (or)
(ii) each vertex of C_n has degree atleast 3 in G.

Proof.

Assume one of the conditions (i) and (ii) holds.
By Theorem 1.3, $\gamma_{cctd}(G) \geq p - 2$. By (i) or (ii), there exists no edge in C_n such that each of its vertices is of degree 2 in G.
Hence, $\gamma_{cctd}(G) \neq p - 2$ and therefore, $\gamma_{cctd}(G) = p - 1$.
Conversely, assume $\gamma_{cctd}(G) = p - 1$.
Let D be a γ_{cctd}-set of G. Then <V − D> ≅ K_1.
If condition (i) and (ii) are not true in G, then $\gamma_{cctd}(G) = p - 2$. □

Theorem 1.6.

Let G be a connected unicyclic graph with the cycle C_n, n ≥ 3. Then, $\gamma_{cctd}(G) = \gamma_{ctd}(G)$ if and only if G is C_n (n ≥ 3) or G is a graph obtained from C_3 by attaching atleast one pendant edge at exactly one vertex of C_3.

Proof.

Let G be a connected unicyclic graph having p ($p \geq 4$) vertices with the cycle C_n, $n \geq 3$.
Assume $\gamma_{cctd}(G) = \gamma_{ctd}(G)$.
For the unicyclic graphs, $\gamma_{cctd}(G) = p - 1$ or $p - 2$.
Therefore, $\gamma_{ctd}(G) = p - 1$ or $p - 2$.
Let $\gamma_{ctd}(G) = p - 1$.
Then, G is a star on p vertices. But, G contains a cycle.
Therefore, $\gamma_{ctd}(G) = p - 2$.
Since G contains a cycle by Theorem ??, G is one of the following graphs.
C_p, K_p (or) G is the graph obtained from a complete graph by attaching pendant edges at atleast one of the vertices of the complete graph.
But G is unicyclic. Therefore, G is the graph obtained from C_3 by attaching pendant edges at atleast one of the vertices of the cycle C_3.
If G is the graph obtained from C_3 by attaching pendant edges at atleast two vertices of C_3, then $\gamma_{cctd}(G) = p - 1$.
Hence, G is one of the graphs given in the theorem.
Conversely, if G is C_n ($n \geq 3$) or G is a graph obtained from C_3 by attaching atleast one pendant edge at exactly one vertex of C_3, then $\gamma_{cctd}(G) = \gamma_{ctd}(G)$. □

In the following, $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$ are found for the unicyclic graphs.

Theorem 1.7.

Let G be a connected unicyclic graph having p ($p \geq 4$) vertices with the unique cycle C_3. Then, $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$ if and only if G is one of the following graphs.

(i) G is obtained from C_3 by attaching pendant edges at atleast 2 vertices of C_3.
(ii) G is obtained from C_3 by attaching a path of length atleast 2 at a vertex, say v of C_3.
(iii) G is obtained from C_3 by joining $K_{1,n}$ ($n \geq 2$) at a vertex, say v of C_3.
(iv) G is the graph obtained from C_3 by attaching a path P of length atleast 3 at a vertex of C_3 and then attaching pendant edges at the support and the pendant vertex of P.
(v) G is obtained from the graph mentioned in (ii), (iii) or (iv) by attaching atleast two pendant edges at v.

Proof.

Let G be a connected unicyclic graph having p ($p \geq 4$) vertices with the cycle C_3. Assume $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$.
By Theorems ?? and ??, $\gamma_{cctd}(G) = p - 1$ or $p - 2$.
Therefore, $\gamma_{ctd}(G) = p - 2$ or $p - 3$.
Case 1. $\gamma_{cctd}(G) = p - 1$.
Therefore, $\gamma_{ctd}(G) = p - 2$.
But, $\gamma_{ctd}(G) = p - 2$ if and only if G is isomorphic to one of the following graphs:
K_p, C_p or G is the graph obtained from a complete graph by attaching pendant edges at atleast one of the vertices of the complete graph (by Theorem ??). Since G is unicyclic, G is either C_p (or) G is the graph obtained from C_3 by attaching pendant edges at atleast one of the vertices of C_3.
If pendant edges are attached at exactly one vertex, then $\gamma_{cctd}(G) = \gamma_{ctd}(G)$.
Therefore, G is the graph obtained from C_3 by attaching pendant edges at atleast two vertices of C_3.
Case 2. $\gamma_{cctd}(G) = p - 2$.
By Theorem ??, $\gamma_{cctd}(G) = p - 2$ if and only if there exists atleast two adjacent vertices of C_n which are of degree 2 in G. Therefore, G is the graph obtained from C_3 by attaching trees at exactly one vertex of C_3.
If either G has an induced path P of length atleast 3 such that the degree of each internal vertex of P is atleast 3 in G and the end vertices of P are not the pendant vertices of G (or) if G has $K_{1,n}$ ($n \geq 3$) as an induced subgraph such that degrees of pendant vertices of $K_{1,n}$ in G are atleast 2 and the degree of central vertex of $K_{1,n}$ is atleast n, then $V(G) - V(P)$ or $V(G) - V(K_{1,n})$ is a ctd-set of G and hence $\gamma_{ctd}(G) \leq p - 4$.
Therefore, G is one of the graphs given in (ii), (iii), (iv) and (v).
Conversely, if G is the graph given as in (i), then $\gamma_{ctd}(G) = p - 2$, whereas $\gamma_{cctd}(G) = p - 1$. For the graphs G mentioned in (ii), (iii), (iv) and (v), $\gamma_{ctd}(G) = p - 3$ and $\gamma_{cctd}(G) = p - 2$. □

In a similar manner, the following theorem can be proved.

Theorem 1.8.

Let G be a connected unicyclic graph with the cycle C_4, then, $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$ if and only if G is one of the following graphs.

(i) G is obtained from C_4 by attaching pendant edges at a vertex or any two adjacent vertices of C_4.
(ii) G is obtained from C_4 by attaching a path of length atleast 2 at a vertex, say v of C_4.
(iii) G is obtained from C_4 by joining $K_{1,n}$ ($n \geq 2$) at a vertex, say v of C_4.
(iv) G is obtained from C_4 by attaching a path P of length atleast 3 at a vertex of C_4 and then attaching pendant edges at the pendant vertex (or) the support of P (or) at both.
(v) G is obtained from C_4 by attaching a path of length 2 and then attaching pendant edges at a pendant vertex of the path.

Theorem 1.9.

Let G be a connected unicyclic with the cycle C_n ($n \geq 5$). Then, $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$ if and only if G is one of the following graphs.

(i) G is obtained from C_n ($n \geq 5$) by attaching pendant edges at atleast one vertex of C_n such that distance between any two vertices of C_n, which are the supports of G, is atleast two and the vertices of C_n, which are not the supports of G, are not independent.
(ii) G is obtained from C_n ($n \geq 5$) by attaching trees (which are not stars) to exactly one vertex, say v of C_n such that the vertices of the trees which are adjacent to v are not the supports of the trees (or G).
(iii) G is obtained from C_n by joining $K_{1,n}$ ($n \geq 1$) to exactly one vertex of C_n.

Proof.

Let G be any connected unicyclic graph with the cycle C_n ($n \geq 5$) as the unique cycle having p vertices such that $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$.
If $G \cong C_n$, then $\gamma_{cctd}(G) = \gamma_{ctd}(G)$. Hence, G has atleast one support
(1) Assume G is the graph obtained from C_n ($n \geq 5$) by attaching pendant edges at vertices of C_n. If any two adjacent vertices of C_n are supports of G, then $\gamma_{ctd}(G) = p - 4$ and $\gamma_{cctd}(G) = p - 2$.
Hence, no two adjacent vertices of C_n are supports of G.
If the vertices of C_n, which are not the supports of G, are independent, then G is the graph obtained from C_n by attaching pendant edges at vertices of C_n which are at distance two. For this graph G, $\gamma_{ctd}(G) = p - 3$ and $\gamma_{cctd}(G) = p - 1$. Therefore, the set of vertices of C_n, which are not the supports of G, are not independent.
In this case, number of supports of G is $\leq \left\lfloor \frac{n-1}{2} \right\rfloor$.
Hence, G is the graph given in (1).
(2) Assume G is the graph obtained from C_n ($n \geq 5$) by attaching trees at any vertex of C_n. If trees are attached at atleast two vertices of C_n (or) trees together with pendant edges are attached at atleast one vertex, then also $\gamma_{cctd}(G) \geq \gamma_{ctd}(G) + 2$.
Therefore, G is the graph obtained from C_n by attaching trees at exactly one vertex, say v of C_n.
If a vertex of the tree adjacent to v is a support and if there exists atleast one nonpendant vertex adjacent to this support, then also $\gamma_{cctd}(G) \geq \gamma_{ctd}(G) + 2$.
Hence, vertices of the tree adjacent to v is not a support (or)
If a vertex of the tree adjacent to v is a support, then there exists no nonpendant vertex adjacent to this support.
Therefore, we have the graphs given in (i), (ii) (or) (iii).
Conversely, if G is one of the graphs given in (i), (ii) (or) (iii),
then $\gamma_{cctd}(G) = \gamma_{ctd}(G) + 1$. □

Theorem 1.10.

There exists a connected graph G such that $\gamma_{cctd}(G) = \gamma_{ctd}(G) + a$, where a is an integer ($a \geq 2$).

Proof.

Consider a cycle C_n ($n \geq 5$) on n vertices.
Let P_{a-1} be a path on $(a-1)$ vertices in C_n, $2 \leq a \leq n - 2$.
Attach a pendant edge and a path of length 2 at one of the end vertices of P_{a-1} and attach exactly one pendant edge at each of the remaining $(a-2)$ vertices. The resulting graph G has $\gamma_{cctd}(G) = \gamma_{ctd}(G) + a$. □

REFERENCES

1. *Frank Harary, Graph Theory, Narosa Publishing House, Reprint, 2001.*
2. *T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker Inc., New York, (1998).*
3. *S. Muthammai, M. Bhanumathi and P. Vidhya, Complementary tree domination in graphs, International Mathematical Forum, 6(26) (2011), 1273–1283.*
4. *O. Ore, Theory of Graphs, Amer. Math Soc. Colloq. Publ., 38, Providence, (1962).*
5. *L. Volkman, On graph with equal domination and covering numbers, Discrete Math., 51 (1994), 211–217.*