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THE METHODS FOR SOLVING PROBLEMS OF CIRCULAR MEMBRANE OSCILLATIONS

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SUMMARY
This work shows solutions to several specific examples of vibrations of circular membranes by the method of separation of variables using Bessel functions.
KEY WORDS: *Bessel equation and function, vibration equation of a circular membrane, mixed problem, Fourier method.*

INTRODUCTION

From the general theory it is known that the solution of the mixed problem for the wave equation by the method of separation of variables was reduced to the Sturm–Liouville problem. If the area in which we construct a solution to the mixed problem is a segment (vibrations of a string), a rectangle (oscillations of a rectangular membrane), then the corresponding Sturm–Liouville problem is posed for a linear differential equation with constant coefficients. If the area in which we are looking for a solution to a mixed problem is a circle, a cylinder, or a ball, then the corresponding Sturm–Liouville problem is posed for a linear equation with variable coefficients. A characteristic feature of these equations is that the coefficient of the highest derivative vanishes at at least one point of the domain boundary. One such equation is the Bessel equation (equation of cylindrical functions)

$$y''(\xi) + \frac{1}{\xi}\, y'(\xi) + \left(1 - \frac{n^2}{\xi^2}\right) y(\xi) = 0, \qquad n = \text{const} \geq 0, \tag{1}$$

and the Bessel function $J_n(\xi)$ with index n, is a solution to the differential equation (1), limited at zero with its first derivative, i.e.

$$|J_n(0)| < \infty, \qquad |J_n'(0)| < \infty.$$

We present without proof the basic properties of the functions $J_n(\xi)$:

I. For any fixed value, $n = 0, 1, 2, \ldots$ the Bessel function $J_n(\xi)$ generates $[0; R]$ a complete orthogonal system on the interval $\left\{J_n\left(\mu_k^{(n)} \frac{\xi}{R}\right)\right\}$ $k = 1, 2, \ldots$ with weight $\rho(\xi) = \xi$:

$$\int_0^R \xi\, J_n\left(\mu_k^{(n)} \frac{\xi}{R}\right) J_n\left(\mu_l^{(n)} \frac{\xi}{R}\right) d\xi = \begin{cases} 0, & k \neq l, \\ \dfrac{R^2}{2} J_{n+1}^2(\mu_k^{(n)}), & k = l, \end{cases} \tag{2}$$

where are $\mu_1^{(n)}, \mu_2^{(n)}, \ldots -$ the zeros of the function $J_n(\xi)$.

II. Any function $f(\xi)$ that is piecewise smooth on an interval $[0; R]$ and satisfies the boundary conditions is expanded into a convergent Fourier series according to the system of Bessel functions for each fixed $n = 0, 1, 2, \ldots$, i.e.

$$f(\xi) = \sum_{k=1}^{+\infty} a_k J_n\left(\mu_k^{(n)} \frac{\xi}{R}\right), \tag{3}$$

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where

$$a_k = \frac{\int_0^R \xi\, f(\xi) J_n\left(\mu_k^{(n)} \frac{\xi}{R}\right) d\xi}{\int_0^R \xi\, J_n^2\left(\mu_k^{(n)} \frac{\xi}{R}\right) d\xi} = \frac{2}{R^2 J_{n+1}^2(\mu_k^{(n)})} \cdot \int_0^R \xi\, f(\xi) J_n\left(\mu_k^{(n)} \frac{\xi}{R}\right) d\xi. \quad (4)$$

1–task. Find a solution to a mixed problem

$$u_{tt} = u_{xx} + \frac{1}{x} u_x + f(t) J_0(\mu_k x), \quad (1.1)$$

where is $\mu_k -$ the positive root of the equation $J_0(\mu) = 0, \quad 0 < x < 1,$

$$u\big|_{x=1} = u\big|_{t=0} = u_t\big|_{t=0} = 0, \quad \left| u\big|_{x=0} \right| < \infty,$$

if $f(t) = t^2 + 1.$

Solution. Since we have zero boundary and initial conditions, we will look for a solution to this problem in the form of a series

$$u(x,t) = \sum_{l=1}^{+\infty} T_l(t) J_0(\mu_l x) \quad (1.2)$$

by the eigenfunctions of the corresponding homogeneous problem, where $T_l(t) -$ the unknown functions are , and $J_0(\mu_l x) -$ the eigenfunctions of the problem of oscillations of a circular membrane are:

$$y'' + \frac{1}{x} y' + \lambda^2 y = 0, \qquad y\big|_{x=1} = 0. \quad (1.3)$$

Substituting (1.2) into equation (1.1), and assuming the validity of term by term differentia-tion of the series, moving all terms to the left, we obtain

$$\sum_{l=1}^{+\infty} \left[T_l''(t) + \mu_l^2 T_l(t) \right] J_0(\mu_l x) = (t^2 + 1) J_0(\mu_k x).$$

From here we find that when $l \neq k$

$$T_l''(t) + \mu_l^2 T_l(t) = 0, \quad (1.4)$$

and when $l = k$

$$T_k''(t) + \mu_k^2 T_k(t) = t^2 + 1. \quad (1.5)$$

From the initial conditions we have

$$T_l(0) = 0, \qquad T_l'(0) = 0, \qquad l = 1, 2, \ldots \quad (1.6)$$

Then the corresponding solution to the Cauchy problem for the homogeneous equation (1.4) under zero conditions (1.6) is only a trivial function $T_l(t) \equiv 0, \quad l \neq k.$ When $l = k$ we obtain the following Cauchy problem:

$$T_k''(t) + \mu_k^2 T_k(t) = t^2 + 1; \qquad T_k(0) = 0, \qquad T_k'(0) = 0. \quad (1.7)$$

The general solution to equation (1.7) has the form:

$$T_k(t) = A_k \cos \mu_k t + B_k \sin \mu_k t + T_k^*(t),$$

where is $T_k^*(t) = a t^2 + b t + c -$ the private solution. Then from

$$2a + \mu_k^2 (a t^2 + b t + c) = t^2 + 1$$

should

$$a = \frac{1}{\mu_k^2}, \quad b = 0, \quad 2a + c\mu_k^2 = 1 \quad \Rightarrow \quad c = \frac{1}{\mu_k^2}\left(1 - \frac{2}{\mu_k^2}\right) = \frac{\mu_k^2 - 2}{\mu_k^4}.$$

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Thus,

$$T_k(t) = A_k \cos \mu_k t + B_k \sin \mu_k t + \frac{1}{\mu_k^2}\left(t^2 + \frac{\mu_k^2 - 2}{\mu_k^4}\right).$$

From the initial conditions (1.7) we have

$$T_k(0) = A_k + \frac{\mu_k^2 - 2}{\mu_k^4} = 0 \quad \Rightarrow \quad A_k = -\mu_k^{-4}(\mu_k^2 - 2),$$

$$T_k'(0) = \mu_k B_k = 0 \quad \Rightarrow \quad B_k = 0.$$

Finally, we can write the solution to the original problem in the form:

$$u(x,t) = \left[\mu_k^{-4}(\mu_k^2 - 2)(1 - \cos \mu_k t) + \mu_k^{-2} t^2\right] J_0(\mu_k x).$$

2–task. Find a solution to a mixed problem

$$u_{tt} = u_{xx} + \frac{1}{x} u_x, \quad 0 < x < 1,$$

$$\left| u\big|_{x=0} \right| < \infty, \qquad u\big|_{x=1} = g(t), \qquad u\big|_{t=0} = u_0(x), \qquad u_t\big|_{t=0} = u_1(x),$$

if:

$$g(t) = \sin^2 t, \qquad u_0(x) = \frac{1}{2}\left[1 - \frac{J_0(2x)}{J_0(2)}\right], \qquad u_1(x) = 0.$$

Solution. In this problem, heterogeneity enters only into the boundary condition. Therefore, in order to transform the problem to homogeneous boundary conditions, we introduce a new required function

$$u(x,t) = \upsilon(x,t) + \omega(x,t), \tag{2.1}$$

where $\omega(x,t)$ we construct the function so that it satisfies the inhomogeneous boundary condition, i.e.

$$\omega\big|_{x=1} = \sin^2 t = \frac{1}{2}(1 - \cos 2t). \tag{2.2}$$

Then the function $\upsilon(x,t)$ will satisfy the homogeneous boundary condition.

The function $\omega(x,t)$ in the form of a series

$$\omega(x,t) = \sum_{k=0}^{+\infty} (A_k \cos kt + B_k \sin kt) J_0(kx) \tag{2.3}$$

by Bessel functions. Substituting $\omega(x,t)$ into the boundary condition, we find

$$\omega\big|_{x=1} = \sum_{k=0}^{+\infty} (A_k \cos kt + B_k \sin kt) J_0(k) = \frac{1}{2}(1 - \cos 2t),$$

$$B_k = 0, \quad k = 0, 1, \ldots, \quad A_0 = \frac{1}{2}, \quad A_2 = -\frac{1}{2J_0(2)}, \quad A_k = 0, \quad \forall k \neq 0;\ 2.$$

Thus,

$$\omega = \omega(x,t) = \frac{1}{2} - \frac{J_0(2x)}{2J_0(2)} \cos 2t. \tag{2.4}$$

Now, taking into account (2.4) , we transform this problem to the new desired function $\upsilon(x,t)$:

$$\upsilon_{tt} + 2\frac{J_0(2x)}{J_0(2)} \cos 2t = \upsilon_{xx} + \frac{1}{x}\upsilon_x - \frac{\cos 2t}{2J_0(2)}\left[J_0''(2x) + \frac{1}{x} J_0'(2x)\right] =$$

$$= \upsilon_{xx} + \frac{1}{x}\upsilon_x + 2\frac{J_0(2x)}{J_0(2)} \cos 2t, \quad \text{those}$$

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$$\upsilon_{tt} = \upsilon_{xx} + \frac{1}{x}\upsilon_x, \quad 0 < x < 1, \tag{2.5}$$

$$\upsilon\,|_{x=1} = 0, \qquad \upsilon\,|_{t=0} = u\,|_{t=0} - \frac{1}{2} + \frac{J_0(2x)}{2J_0(2)}\cos 2t\,|_{t=0} =$$

$$= \frac{1}{2}\left[1 - \frac{J_0(2x)}{2J_0(2)}\right] - \frac{1}{2}\left[1 - \frac{J_0(2x)}{2J_0(2)}\right] = 0,$$

$$\upsilon_t\big|_{t=0} = u_t\big|_{t=0} - \frac{J_0(2x)}{J_0(2)}\sin 2t\,|_{t=0} = 0.$$

The solution to the homogeneous equation about free vibrations of a circular membrane (2.5) under zero boundary and initial conditions is only a zero function, i.e. $\upsilon(x,t) \equiv 0$.

Consequently, the solution to the mixed problem will be written in the form

$$u(x,t) = \omega(x,t) = \frac{1}{2}\left[1 - \frac{J_0(2x)}{J_0(2)}\cos 2t\right].$$

3–task. Solve a mixed problem

$$u_{xx} + \frac{1}{x}u_x = u_{tt} + u, \quad 0 < x < 1, \tag{3.1}$$

$$\left|u\,|_{x=0}\right| < \infty, \qquad u\,|_{x=1} = \cos 2t + \sin 3t, \qquad u\,|_{t=0} = \frac{J_0\left(x\sqrt{3}\right)}{J_0\left(\sqrt{3}\right)}, \qquad u_t\big|_{t=0} = \frac{3J_0\left(2x\sqrt{2}\right)}{J_0\left(2\sqrt{2}\right)}.$$

Solution. As in the previous example, when the inhomogeneity is contained both in the equation and in the boundary condition, we look for a solution to this problem in the form of a sum

$$u(x,t) = \upsilon(x,t) + \omega(x,t), \tag{3.2}$$

where $\omega(x,t)$ we construct the function so that it satisfies the inhomogeneous boundary condition, i.e.

$$\omega\,|_{x=1} = \cos 2t + \sin 3t. \tag{3.3}$$

Further, as usual in the Fourier method, we formally assume

$$u(x,t) = X(x)T(t) \tag{3.4}$$

and find partial solutions that satisfy the homogeneous boundary condition.

Differentiating function (3.4), then substituting (3.1) into this equation and separating the variables, we obtain:

$$\frac{T''(t)}{T(t)} = \frac{X''(x) + \frac{1}{x}X'(x) - X(x)}{X(x)} = -\lambda^2. \tag{3.5}$$

Equalities (3.5) are reduced to two equations:

$$T''(t) + \lambda^2 T(t) = 0, \tag{3.6}$$

$$X''(x) + \frac{1}{x}X'(x) + (\lambda^2 - 1)X(x) = 0. \tag{3.7}$$

One of the particular solutions to equation (3.7) in the notation $\mu^2 = \lambda^2 - 1$ is the function

$$J_0(\mu x) = J_0\left(x\sqrt{\lambda^2 - 1}\right). \tag{3.8}$$

Thus, it becomes clear that the function $\omega(x,t)$ can be constructed as a series:

$$\omega(x,t) = \sum_{k=1}^{+\infty}(A_k \cos kt + B_k \sin kt)J_0\left(x\sqrt{k^2 - 1}\right). \tag{3.9}$$

From the boundary condition (3.3) it follows that

$$\sum_{k=1}^{+\infty}(A_k\cos kt+B_k\sin kt)J_0\left(\sqrt{k^2-1}\right)=\cos 2t+\sin 3t.$$

From here

$$A_2J_0\left(\sqrt{3}\right)=1,\quad B_3\sqrt{8}=1.\quad A_k=0,\quad \forall k\neq 2;\quad B_k=0,\quad \forall k\neq 3.$$

Then

$$\omega(x,t)=\frac{J_0\left(x\sqrt{3}\right)}{J_0\left(\sqrt{3}\right)}\cos 2t+\frac{J_0\left(2x\sqrt{2}\right)}{J_0\left(2\sqrt{2}\right)}\sin 3t. \tag{3.10}$$

In our case, in the transformation

$$u(x,t)=\upsilon(x,t)+\omega(x,t)$$

the function $\omega(x,t)$ was found in such a way that it satisfies not only the boundary condition, but also the equation, and also, as is easy to see, the initial conditions. Therefore, the solution to this problem is the function

$$u(x,t)=\omega(x,t)=\frac{J_0\left(x\sqrt{3}\right)}{J_0\left(\sqrt{3}\right)}\cos 2t+\frac{J_0\left(2x\sqrt{2}\right)}{J_0\left(2\sqrt{2}\right)}\sin 3t.$$

4–task. Solve a mixed problem

$$u_{tt}=u_{xx}+\frac{1}{x}u_x-\frac{u}{x^2},\qquad 0<x<1, \tag{4.1}$$

$$\left|u\big|_{x=0}\right|<\infty,\qquad u\big|_{x=1}=\sin 2t\cos t,\qquad u\big|_{t=0}=0,\qquad u_t\big|_{t=0}=\frac{J_1(x)}{2J_1(1)}+\frac{3}{2}\frac{J_1(3x)}{J_1(3)}.$$

Solution. First, let's translate the inhomogeneity in the boundary condition into an equation; to do this, we'll construct a function $\omega=\omega(x,t)$ that satisfies the boundary condition , i.e.

$$\omega\big|_{x=1}=\sin 2t\cos t=\frac{1}{2}\sin t+\frac{1}{2}\sin 3t. \tag{4.2}$$

The function $J_1(\mu x)$ satisfies the equation

$$J_1''(\mu x)+\frac{1}{x}J_1'(\mu x)-\frac{1}{x^2}J_1(\mu x)=-\mu^2J_1(\mu x), \tag{4.3}$$

therefore $\omega(x,t)$ can be constructed in the form of the following series

$$\omega(x,t)=\sum_{k=0}^{+\infty}(A_k\cos kt+B_k\sin kt)J_1(kx) \tag{4.4}$$

by Bessel functions. Then from the boundary condition (4.2) it follows that

$$\sum_{k=0}^{+\infty}(A_k\cos kt+B_k\sin kt)J_1(k)=\frac{1}{2}\sin t+\frac{1}{2}\sin 3t,$$

$$A_k=0,\quad k=0,1,\dots,\quad B_1J_1(1)=\frac{1}{2},\quad B_3J_1(3)=\frac{1}{2},\quad B_k=0,\quad \forall k\neq 1;\,3.$$

Thus,

$$\omega(x,t)=\frac{J_1(x)}{2J_1(1)}\sin t+\frac{3}{2}\cdot\frac{J_1(3x)}{J_1(3)}\sin 3t. \tag{4.5}$$

Now, taking into account (4.5), we transform this problem to the new desired function: $\upsilon(x,t)$, i.e. we'll make a replacement

$$u(x,t)=\upsilon(x,t)+\omega(x,t) \tag{4.6}$$

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$$\upsilon_{tt} - \frac{J_1(x)}{2J_1(1)}\sin t - \frac{9}{2}\cdot\frac{J_1(3x)}{J_1(3)}\sin 3t =$$
$$= \upsilon_{xx} + \frac{1}{x}\upsilon_x - \frac{\upsilon}{x^2} + \left[J_1''(x) + \frac{1}{x}J_1'(x) - \frac{1}{x^2}J_1(x)\right]\cdot\frac{\sin t}{2J_1(1)} +$$
$$+ \left[J_1''(3x) + \frac{1}{x}J_1'(3x) - \frac{1}{x^2}J_1(3x)\right]\cdot\frac{\sin 3t}{2J_1(3)} =$$
$$= \upsilon_{xx} + \frac{1}{x}\upsilon_x - \frac{\upsilon}{x^2} - \frac{J_1(x)}{2J_1(1)}\sin t - \frac{9}{2}\cdot\frac{J_1(3x)}{J_1(3)}\sin 3t,$$

from here

$$\upsilon_{tt} = \upsilon_{xx} + \frac{1}{x}\upsilon_x - \frac{\upsilon}{x^2}, \quad 0 < x < 1, \tag{4.7}$$
$$\upsilon\big|_{x=1} = 0, \qquad \upsilon\big|_{t=0} = 0, \qquad \upsilon_t\big|_{t=0} = 0. \tag{4.8}$$

The solution to the homogeneous equation (4.7) under zero initial and boundary conditions (4.8) is only a trivial function, i.e. $\upsilon(x,t) \equiv 0.$

Thus, the solution to this problem is

$$u(x,t) = \frac{J_1(x)}{2J_1(1)}\sin t + \frac{J_1(3x)}{2J_1(3)}\sin 3t.$$

Let us now consider a somewhat more general problem about forced oscillations of a circular membrane of radius R, fixed along the edge.

It is known that the eigenvalues and eigenfunctions of the Laplace operator $-\Delta_{r,\varphi}$ in a circle of radius R, i.e. next boundary value problem

$$-\Delta_{r,\varphi}u = \lambda u \quad \Rightarrow \quad -\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}\right) = \lambda u; \qquad u\big|_{r=R} = 0,$$

are defined as follows:

$$\text{at} \quad \lambda = \left(\frac{\mu_k^{(0)}}{R}\right)^2, \quad k = 1, 2, \ldots \ \text{and} \quad Y_0^k(r,\varphi) = J_0\left(\mu_k^{(0)}\frac{r}{R}\right)\cdot 1,$$

$$\text{at} \quad \lambda = \left(\frac{\mu_k^{(n)}}{R}\right)^2, \quad k = 1, 2, \ldots \ \text{and} \quad Y_n^k(r,\varphi) = \begin{cases} J_n\left(\mu_k^{(n)}\frac{r}{R}\right)\cdot\cos n\varphi, \\ J_n\left(\mu_k^{(n)}\frac{r}{R}\right)\cdot\sin n\varphi. \end{cases}$$

5–task. Solve a mixed problem

$$u_{tt} = \Delta u + J_2(\mu_4^{(2)}r)\cos 2\varphi\cos(\mu_4^{(2)}t), \qquad r < R, \tag{5.1}$$
$$u\big|_{r=1} = 0, \qquad u\big|_{t=0} = f(r)\sin\varphi, \qquad u_t\big|_{t=0} = J_2(\mu_4^{(2)}r)\cos 2\varphi, \tag{5.2}$$

where $u = u(r,\varphi,t), \qquad \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}.$

Solution. All eigenfunctions form a complete orthogonal system, then the function $u(r,\varphi,t)$ in the domain $\Omega = \{r < R,\ 0 < \varphi < 2\pi,\ t > 0\}$ can be represented as

$$u(r,\varphi,t) = \sum_{k=1}^{+\infty} J_0\left(\mu_k^{(0)}\frac{r}{R}\right)\cdot 1\cdot T_{0,k}(t) +$$

$$+\sum_{n=1}^{+\infty}\left[\sum_{k=1}^{+\infty}J_n\left(\mu_k^{(n)}\frac{r}{R}\right)\cdot\cos n\varphi\cdot T_{n,k}(t)+\sum_{k=1}^{+\infty}J_n\left(\mu_k^{(n)}\frac{r}{R}\right)\cdot\sin n\varphi\cdot\tilde{T}_{n,k}(t)\right]. \tag{5.3}$$

Now we expand the inhomogeneity in the equation and in the initial conditions into a series according to the eigenfunctions of the operator Δ in the region $\Omega,\ R=1.$

Let us first note that $J_2(\mu_4^{(2)}r)\cos 2\varphi$ is one of the eigenfunctions $J_n(\mu_k^{(n)}r)\cos n\varphi$, then we write out the expansion

$$f(r)=\sum_{k=1}^{+\infty}a_k J_1(\mu_k^{(1)}r), \tag{5.4}$$

since $\sin\varphi$ it is a one-time simple function. From here

$$a_k=\frac{\int_0^1 f(r)J_1(\mu_k^{(1)}r)\,r\,dr}{\int_0^1 J_1^2(\mu_k^{(1)}r)\,r\,dr}=\frac{1}{\left\|J_1(\mu_k^{(1)}r)\right\|^2}\cdot\int_0^1 f(r)J_1(\mu_k^{(1)}r)\,r\,dr, \tag{5.5}$$

where $\left\|J_1(\mu_k^{(1)}r)\right\|^2=\int_0^1 J_1^2(\mu_k^{(1)}r)\,r\,dr=\frac{1}{2}\left[J_1'(\mu_k^{(1)})\right]^2.$

Thus, taking into account these expansions in terms of the eigenfunctions of the operator Δ, it is enough to look for a solution to the problem of forced oscillations of a circular membrane of radius 1, fixed along the edge in the form:

$$u(r,\varphi,t)=J_2(\mu_4^{(2)}r)\cos 2\varphi\cdot T_{2,4}(t)+\sum_{k=1}^{+\infty}J_1(\mu_k^{(1)}r)\sin\varphi\cdot\tilde{T}_{1,k}(t). \tag{5.6}$$

Substituting the last expression into this equation and into the initial conditions, we get:

$$\begin{aligned}J_2(\mu_4^{(2)}r)\cos 2\varphi\cdot T_{2,4}''(t)+\sum_{k=1}^{+\infty}J_1(\mu_k^{(1)}r)\sin\varphi\cdot\tilde{T}_{1,k}''(t)=\\ =-(\mu_4^{(2)})^2J_2(\mu_4^{(2)}r)\cos 2\varphi\cdot T_{2,4}(t)-\\ -\sum_{k=1}^{+\infty}(\mu_k^{(1)})^2J_1(\mu_k^{(1)}r)\sin\varphi\cdot\tilde{T}_{1,k}(t)+J_2(\mu_4^{(2)}r)\cos 2\varphi\cos(\mu_4^{(2)}t),\end{aligned} \tag{5.7}$$

and the initial conditions

$$J_2(\mu_4^{(2)}r)\cos 2\varphi\cdot T_{2,4}(0)+\sum_{k=1}^{+\infty}J_1(\mu_k^{(1)}r)\sin\varphi\cdot\tilde{T}_{1,k}(0)=\sum_{k=1}^{+\infty}a_kJ_1(\mu_k^{(1)}r)\sin\varphi,$$

$$J_2(\mu_4^{(2)}r)\cos 2\varphi\cdot T_{2,4}'(0)+\sum_{k=1}^{+\infty}J_1(\mu_k^{(1)}r)\sin\varphi\cdot\tilde{T}_{1,k}(0)=J_2(\mu_4^{(2)}r)\cos 2\varphi.$$

Next, we collect similar terms

$$J_2(\mu_4^{(2)}r)\cos 2\varphi:\quad\begin{cases}T_{2,4}''(t)=-(\mu_4^{(2)})^2T_{2,4}(t)+\cos(\mu_4^{(2)}t),\\ T_{2,4}(0)=0,\quad T_{2,4}'(0)=1,\end{cases} \tag{5.8}$$

$$J_1(\mu_k^{(1)}r)\sin\varphi:\quad\begin{cases}\tilde{T}_{1,k}''(t)=-(\mu_k^{(1)})^2\tilde{T}_{1,k},\\ \tilde{T}_{1,k}(0)=a_k,\quad \tilde{T}_{1,k}'(0)=0.\end{cases} \tag{5.9}$$

$\tilde{T}_{1,k}(t)-$ the general solution of the homogeneous equation has the form:

$$\tilde{T}_{1,k}(t)=C_1\cos\mu_k^{(1)}t+C_2\sin\mu_k^{(1)}t,$$

$$a_k=\tilde{T}_{1,k}(0)=C_1,\quad 0=\tilde{T}_{1,k}'(0)=C_2\mu_k^{(1)},\quad \mu_k^{(1)}>0\ \Rightarrow\ C_2=0.$$

Thus , $\tilde{T}_{1,k}(t) = a_k \cos(\mu_k^{(1)} t).$

The general solution $T''_{2,4} = -(\mu_4^{(2)})^2 T_{2,4} + \cos(\mu_4^{(2)} t)$ of the inhomogeneous equation has the form:

$$T_{2,4}(t) = A\cos(\mu_4^{(2)} t) + B\sin(\mu_4^{(2)} t) + T^*(t), \qquad (5.10)$$

where a particular solution should be sought in the form

$$T^*(t) = at\cos(\mu_4^{(2)} t) + bt\sin(\mu_4^{(2)} t). \qquad (5.11)$$

$$\frac{d}{dt}: \;\; a\cos(\mu_4^{(2)} t) - at\mu_4^{(2)} \sin(\mu_4^{(2)} t) + b\sin(\mu_4^{(2)} t) + bt\mu_4^{(2)} \cos(\mu_4^{(2)} t),$$

$$\frac{d^2}{dt^2}: \quad -2a\mu_4^{(2)} \sin(\mu_4^{(2)} t) - at(\mu_4^{(2)})^2 \cos(\mu_4^{(2)} t) +$$

$$+ 2b\mu_4^{(2)} \cos(\mu_4^{(2)} t) - bt(\mu_4^{(2)})^2 \sin(\mu_4^{(2)} t) =$$

$$= -at(\mu_4^{(2)})^2 \cos(\mu_4^{(2)} t) - bt(\mu_4^{(2)})^2 \sin(\mu_4^{(2)} t) + \cos(\mu_4^{(2)} t) \;\; \Rightarrow$$

$$\Rightarrow -2a\mu_4^{(2)} = 0 \;\; \Rightarrow \;\; a = 0; \qquad 2b\mu_4^{(2)} = 1 \;\; \Rightarrow \;\; b = \frac{1}{2\mu_4^{(2)}}.$$

Let us substitute the found values a and b into the expression for $T^*(t)$, then

$$T_{2,4}(t) = A\cos(\mu_4^{(2)} t) + B\sin(\mu_4^{(2)} t) + \frac{t}{2\mu_4^{(2)}} \sin(\mu_4^{(2)} t). \qquad (5.12)$$

From the initial conditions we determine the unknown coefficients A and B:

$$T_{2,4}(0) = 0 \quad \Rightarrow \quad A = 0; \qquad T'_{2,4}(0) = 1 \quad \Rightarrow \quad B\mu_4^{(2)} = 1.$$

Then

$$T_{2,4}(t) = \frac{1}{\mu_4^{(2)}} \left(1 + \frac{t}{2}\right) \sin(\mu_4^{(2)} t) \qquad (5.13)$$

and the final answer is:

$$u(r,\varphi,t) = \frac{1}{\mu_4^{(2)}} \left(1 + \frac{t}{2}\right) J_2(\mu_4^{(2)} r) \cdot \cos 2\varphi \cdot \sin(\mu_4^{(2)} t) +$$

$$+ \sum_{k=1}^{+\infty} a_k J(\mu_4^{(2)} r) \cdot \sin\varphi \cdot \cos(\mu_4^{(2)} t),$$

where the coefficient a_k is determined by formula (5.5)

$$a_k = \frac{1}{\left\| J_1(\mu_k^{(1)} r) \right\|^2} \cdot \int_0^1 r\, f(r) J_1(\mu_k^{(1)} r) dr.$$

CONCLUSION

The results obtained can be used in the theory of linear differential and integral operators, in mathematical physics when integrating nonlinear equations.

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