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SIMILARITY OF QUADRILATERALS

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ABSTRACT

In the article, there is very little information about the similarity of rectangles, but its applications are very wide, making it easy to solve olympiad problems. The paper presents the theorem on the similarity of rectangles and the solution of problems using this theorem.

KEYWORDS: *similarity, quadrilateral, cyclic, ratio, theorem, congruent, polar, circle, distance, angle, point, constant, midpoint, parallelogram, circumcircle.*

We know very well about similarity of triangles. Horewer, although used widely, not a lot of people know about similarity of quadrilaterals. The method of finding similar quadrilaterals is called **SSAAA Similarity**. The following theorem will look into it broadly.

Theorem-1 (SSAAA Similarity) Two quadrilaterals $ABCD$ and $A'B'C'D'$ satisfy $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'}$. Prove that $ABCD$ and $A'B'C'D'$ are similar.

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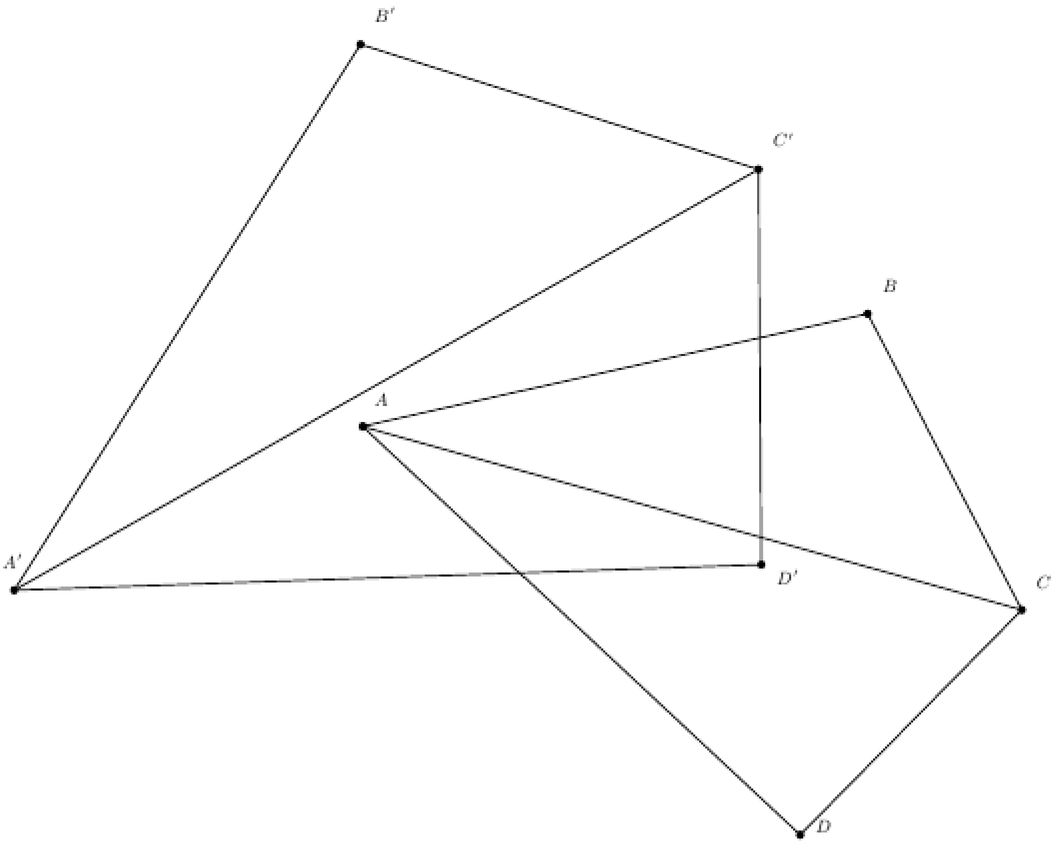
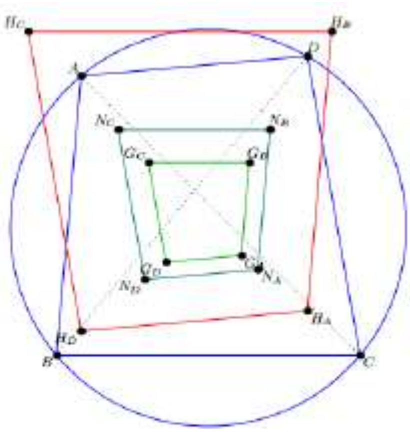


Proof: Notice $\triangle ABC$ and $\triangle A'B'C'$ are similar from SAS similarity. Therefore $\angle C'A'D' = \angle A' - \angle B'A'C' = \angle A - \angle BAC = \angle CAD$. Similarly $\angle A'C'D' = \angle ACD$ so $\triangle A'C'D' \sim \triangle ACD$. Now notice we have $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{AD}{A'D'} = \frac{CD}{C'D'}$ so our proof is complete.

Now let's use this theorem to solve some interesting and hard problems.

Problem-1 . Let $ABCD$ be a cyclic quadrilateral. We define G_A, H_A, N_A to denote the centroid, orthocenter, and nine-point center of $\triangle BCD$. We define G_B, etc. similarly. Show that $ABCD$, $H_AH_BH_CH_D$, $G_AG_BG_CG_D$, $N_AN_BN_CN_D$ are all similar quadrilaterals with a similarity ratio of $6:6:2:3$, respectively.

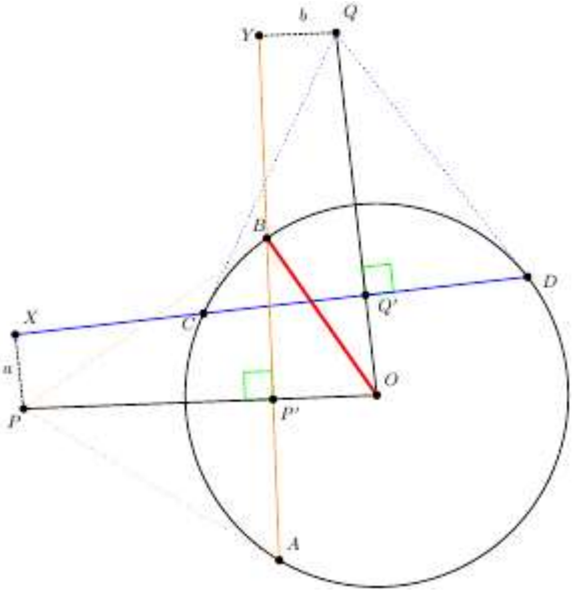


Proof: From a well-known lemma, $AH_B = 2R\cos\angle DAC$. We know

$$2R\cos\angle DAC = 2R\cos\angle DBC = BH_A$$

Also $AH_B \perp DC \perp BH_A$, so ABH_AH_B is a parallelogram, so $AB \parallel H_AH_B$, $AB = H_AH_B$. Similarly, $BC \parallel H_BH_C$, $CD \parallel H_CH_D$, $DA \parallel H_DH_A$, and so $ABCD$ is similar to $H_AH_BH_CH_D$, and furthermore congruent. It is well known that the nine-point center N_A lies on the Euler line, and is the midpoint between the circumcenter O and the orthocenter H_A. Therefore $N_AN_BN_CN_D$ is simply a dilation of $H_AH_BH_CH_D$ by a factor of two, meaning it is similar to $H_AH_BH_CH_D$ and therefore similar to $ABCD$ with a ratio of $\frac{1}{2}$. It is well-known that $OG:OH = 1:3$, so $G_AG_BG_CG_D$ is similar to $H_AH_BH_CH_D$ with a ratio of $\frac{1}{3}$. Since $H_AH_BH_CH_D$ is congruent to $ABCD$, we get our result.

Problem-2 . (Salmon's Lemma) Let O be the center of an arbitrary circle and let P, Q be arbitrary points. If a is the distance from P to the polar of Q, and b is the distance from Q to the polar of P, then show that $\frac{OP}{OQ} = \frac{a}{b}$.



Solution: Draw the tangents from P, Q to circle O to intersect at A, B, C, D respectively. Notice AB is the polar of P, and CD is the polar of Q. Drop perpendiculars from P, Q to CD, AB at X, Y respectively. Let PO intersect AB at P', and QO intersect CD at Q'.

Notice that because AB is the polar of P, we have $\angle PP'B = \angle PBO = 90^\circ$. From similar triangles $\triangle P'BO \sim \triangle BPO$, we have $OP \cdot OP' = OB^2 = R^2$, and similarly from right triangle OCQ we have $OQ \cdot OQ' = OC^2 = R^2$, so $OQ \cdot OQ' = OP \cdot OP'$. Notice quadrilaterals $XPOQ'$ and $YQOP'$ are similar because they have: two right angles each at X, Q' and Y, P'; share an angle at O; and furthermore satisfy $\frac{OQ'}{OP'} = \frac{OP}{OQ}$. From this we get $\frac{OP}{OQ} = \frac{PX}{QY} = \frac{a}{b}$ as desired.

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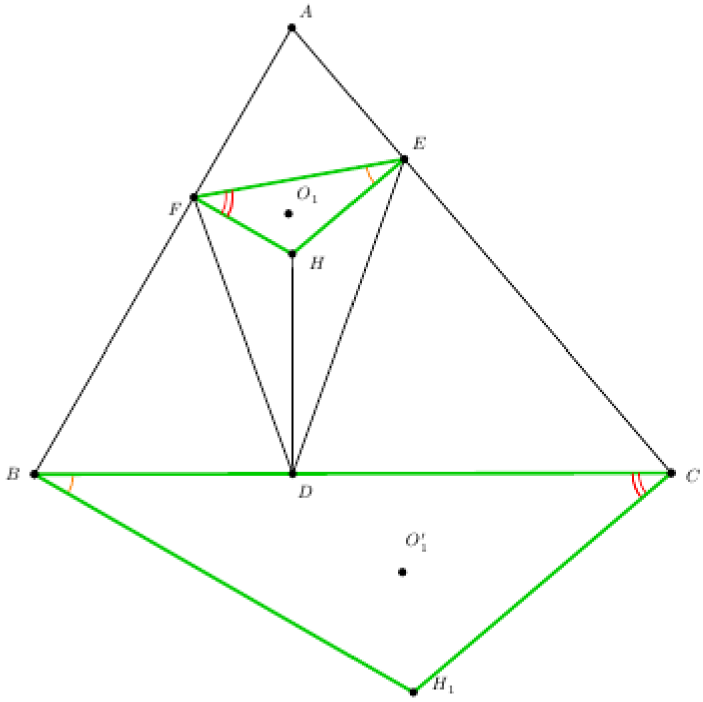
Problem-3. In a triangle ABC, construct altitudes AD, BE, CF and let H be the orthocenter. Let O_1, O_2, O_3 be the incenters of triangles EHF, FHD, DHE, respectively. Prove that the lines AO_1, BO_2, CO_3 are concurrent.



Proof: Denote H_1 as the point outside triangle ABC where $\angle CBH_1 = \angle FEH$ and $\angle BCH_1 = \angle EFH$. Denote the incenter of the triangle H_1BC as $O_1^{'}$. Define $O_2^{'}$ and $O_3^{'}$ similarly. Then, by Jacobi's Theorem, we see that $AO_1^{'}$, $BO_2^{'}$, $CO_3^{'}$ are concurrent at a point P. Notice that because $\angle BFC = \angle BEC = 90^\circ$, $BFEC$ is cyclic so from Power of a Point, $\frac{AF}{AE} = \frac{AC}{AB}$. Also, $\angle AFE = \angle ACB$, $\angle AEF = \angle ABC$ so $\angle AFH = \angle ACH_1$ and $\angle AEH = \angle ABH_1$. So from $SSAAA$ similarity, we know quadrilaterals $AEHF$ and ABH_1C are similar. Therefore $\angle BAO_1 = \angle CAO_1^{'}$. Similarly, we get $\angle CBO_2 = \angle ABO_2^{'}$ and $\angle ACO_3 = \angle BCO_3^{'}$, so AO_1, BO_2, CO_3 are concurrent at the isogonal conjugate of P.

Next, we will prove an intriguing lemma that will be key to a more difficult problem. Both the lemma and the following problem will use properties of similar quadrilaterals in their proofs.

Lemma: Let $A_1A_2...A_n$ and $B_1B_2...B_n$ be directly similar n-gons, and k be a real number. Show that $C_1C_2...C_n$ is similar to both $A_1A_2...A_n$ and $B_1B_2...B_n$, where for $1 \le i \le n$, C_i is on A_iB_i with ratio $A_iC_i : C_iB_i = k$.

Proof: Consider the center of spiral similarity X of $A_1A_2...A_n$ and $B_1B_2...B_n$. Therefore all triangles of the form A_iXB_i for $1 \le i \le n$ are similar. We know all degenerate quadrilaterals of the form $XA_iC_iB_i$ have $A_iC_i : C_iB_i$, $\angle XA_iC_i$, $\angle A_iC_iB_i$ and $\angle XB_iC_i$ constant, because they are equal to k, $\angle XA_iB_i$, 180°, and $\angle XB_iA_i$, respectively, which are known to be constant. Therefore, from $SSAAA$

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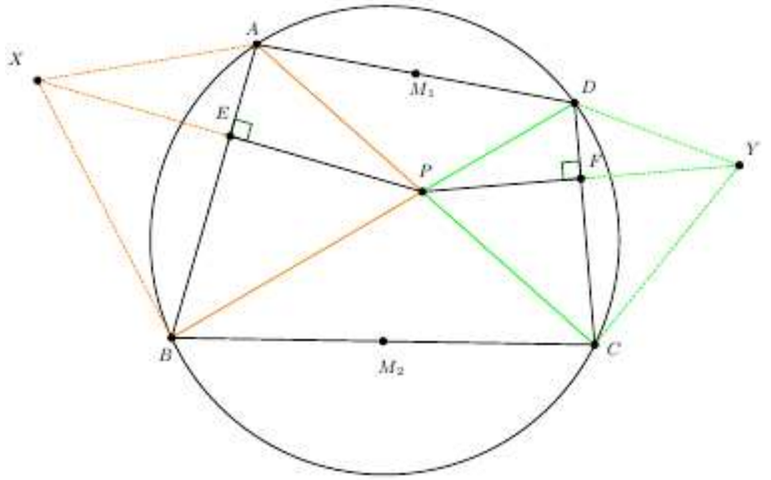
similarity, we have for all $1 \le i \le n$, the quadrilaterals $XA_iC_iB_i$ are similar. It is now clear that the ratio $\frac{XC_i}{XA_i}$ and the angle $\angle C_iXA_i$ are constant over $1 \le i \le n$, meaning there exists a spiral similarity mapping $A_1A_2...A_n$ to $C_1C_2...C_n$ implying the result.

Problem-4. (USA TST – 2000) Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.



Proof: Denote P as the intersection of the diagonals and M_1 and M_2 as the midpoints of AD and BC, respectively. Let X and Y be the reflections of P across AB and CD, respectively. Since $\angle ABD = \angle ACD$ and $\angle BAC = \angle BDC$, triangles APB and DPC are similar, which implies quadrilaterals $APBX$ and $DYCP$ are similar. Notice quadrilateral M_1EM_2F is formed by the midpoints of AD, PY, BC and PX so from the preceding lemma, this quadrilateral is similar to both $APBX$ and $DYCP$. Because $APBX$ and $DYCP$ are kites, M_1EM_2F is also a kite which implies $M_1M_2 \perp EF$ as desired.

Problem-5. (ISL– 2009) Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

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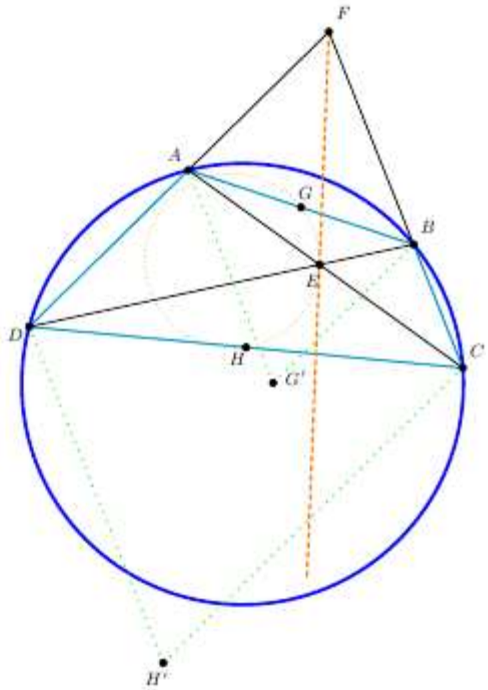

Proof: Let G' be the reflection of point F across point G, and let H' be the reflection of point F across point H. Note that we have parellelograms $FAG'B$ and $FCH'D$, which are similar because triangles FAB and FCD are similar. From this and from $\triangle EAB \sim \triangle EDC$ we obtain that quadrilaterals $FDEC$ and $G'AEB$ are similar, and the quadrilaterals $FAEB$ and $H'DEC$ are similar. These pairs of similar quadrilaterals implies that the position of point E in $FAG'B$ corresponds to the same position as point E in $FCH'D$. Therfore, we have that $\angle G'EF = \angle H'EF$ which implies that E, G', H' are collinear. Because $G'H'$ is a homothetic transformation of scale 2 of GH in respect to the point F, we see that GH and $G'H'$ are parallel. This means that $\angle GHE = \angle H'EH$, and since we know that $\angle FEG = \angle H'EH$, we have $\angle FEG = \angle GHE$ implying EF is a tangent to the circumcircle of GEH at E.

Problem-6. (IMO 2013) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of altitudes from B and C, respectively. Denote by w_1 the circumcircle of $\triangle BWN$, and let X be the point such that WX is a diameter of w_1. Analogously, denote by w_2 the circumcircle of CWM, and let Y be the point such that WY is a diameter of w_2. Prove that X, Y and H are collinear.

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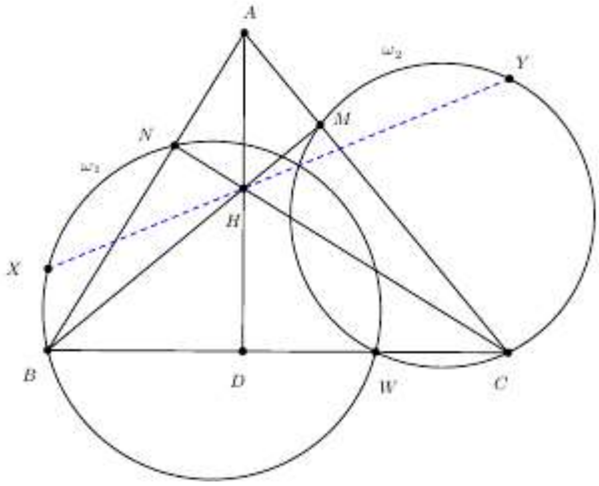
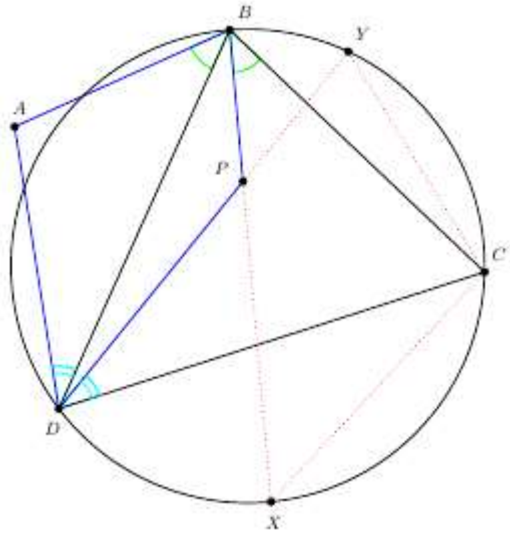


Proof: Note that $\angle ANC = \angle HNB = 90^\circ$, and $\angle A = 90^\circ - \angle ABM = \angle NHB$, so $\triangle NHB \sim \triangle NAC$. Also, we know $\angle NXB = 180^\circ - \angle NWB = \angle NWC$, and because WX is a diameter, $\angle XNB = 90^\circ - \angle BNW = \angle WNC$. Therefore, $\triangle XNB \sim \triangle WNC$. From both similar triangles we get $\angle NAC = \angle NHB$, $\angle NXB = \angle NWC$, $\angle ANW = \angle ANC + \angle CNW = \angle HNB + \angle BNX = \angle XNH$ and $\frac{NA}{AC} = \frac{NH}{HB}$. Therefore, from $SSAAA$, quadrilaterals $XNHB$ and $WNAC$ are similar. From this we obtain $\angle NHX = \angle NAW = \angle WAB$. We analogously obtain quadrilaterals $YMHC$ and $WMAB$ are similar, meaning $\angle WAB = \angle YHC$, and we conclude that $\angle NHX = \angle YHC$. Therefore points X, Y, H are collinear.

Problem-7. (IMO 2004) In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the $\angle ABC$ nor the $\angle CDA$. The point P lies inside $ABCD$ and satisfies $\angle PBC = \angle DBA$, $\angle PDC = \angle BDA$. Prove that $ABCD$ is a cyclic quadrilateral if $AP = CP$.



Proof: Assume $A \notin (BDC)$. Let $X = BP \cap (BDC)$ and $Y = DP \cap (BDC)$. We shall provethat quadrilaterals $PBAD$ and $PYCX$ are similar, and furthermore congruent. By angle-chasing: $\angle ABP = \angle CBD = \angle CYD$, $\angle ADP = \angle CDB = \angle CXB$ and $\angle BPD = \angle YPX$. Also, by Power of a Point $\frac{PB}{PD} = \frac{PY}{PX}$. So from $SSAAA$ similarity we can conclude that $PBAD \sim PYCX$. But the similarity ratio between the two is $\frac{AP}{CP}$ which equals one, so the two quadrilaterals are congruent. In fact they are reflections of each other over the angle bisector of $\angle DPX$. Since $BYCXD$ is cyclic its reflection $YBADX$ is cyclic as well. But these two circles are the same implying that $ABCD$ is cyclic.

Independent Study Problems.

Problem-8. (India – 2014) In a acute-angled triangle ABC, a point D lies on the segment BC. Let O_1O_2 denote the circumcentres of triangles ABD and ACD respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

Problem-9. (Korea – 2010) Let $ABCD$ be a cyclic convex quadrilateral. Let E be the intersection of lines AB, CD. P is the intersection of line passing B and perpendicular to AC, and line passing C and perpendicular to BD. Q is the intersection of line passing D and perpendicular to AC, and line passing A and perpendicular to BD. Prove that three points E, P, Q are collinear.

Problem-10. (USA – 2006) Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

Problem-11. (ISL – 2003) Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Problem-12. (Greece – 2001) A triangle ABC is inscribed in a circle of radius R. Let BD and CE be the bisectors of the angles B and C respectively and let the line DE meet the arc AB not containing C at point K. Let A_1, B_1, C_1 be the feet of perpendiculars from K to BC, AC, AB and x, y be the distances from D and E to BC, respectively.

(a) Express the lengths of KA_1, KB_1, KC_1 in terms of x, y and the ratio $l = \frac{KD}{ED}$.

(b) Prove that $\frac{1}{KB} = \frac{1}{KA} + \frac{1}{KC}$.

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