



MATHEMATICAL MODELING OF NONLINEAR THERMAL PROCESSES

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ABSTRACT

When describing most real physical processes, nonlinear partial differential equations are obtained. The study of the general properties of nonlinear equations and methods for their solution is an urgent direction in the field of computational technology. Having interesting facts and a variety of effective methods for researching and solving nonlinear equations, this area of applied mathematics still does not have such a solid theoretical foundation as the theory of linear equations. Among the systems of nonlinear partial differential equations, the most common are quasilinear equations. But even for these systems at present there is no sufficiently complete theory, there are no general theorems of existence and uniqueness of the solution of the problem. For the numerical solution of quasilinear equations, difference methods or the method of grids are mainly used. It allows one to reduce the solution of a quasilinear partial differential equation to the solution of systems of linear algebraic equations.

KEYWORDS: *Implicit scheme, implicit iteration scheme, number of iterations, number of arithmetic operations, number of grid layers, grid steps, linear and nonlinear difference schemes, thermal conductivity coefficient, quasilinear equation, initial and boundary conditions.*

INTRODUCTION

Many applied problems are described by quasilinear and nonlinear equations of mathematical physics [1-2]. Difference schemes are the main mathematical apparatus for solving such equations [3-4]. The work [5-6] is devoted to the solution of multidimensional problems by the method of fractional steps. Mathematical problems in the theory of systems of quasilinear equations of gas dynamics and difference methods for their solution are presented in [7]. Various approaches to the solution of quasilinear equations are described in [9-13]. In [14], an analytical and numerical study of a one-dimensional boundary value problem with degeneration for a nonlinear heat equation in the case of a power-law dependence of the heat conductivity coefficient on temperature, the solution of which has the form of a heat wave propagating along a cold background with a finite velocity, was carried out. A numerical algorithm based on the boundary element method is applied. The article [15] is devoted to finding invariant solutions of the nonlinear equation of heat conduction without sources and sinks with a power-law dependence of the coefficient of thermal conductivity on temperature. The problem under consideration is reduced to Cauchy problems for ordinary differential equations with a singularity at the highest derivative, for which a theorem on the existence and uniqueness of a solution in the class of analytic functions is proved. In [16], the evolution of a thermal perturbation in a nonlinear medium was studied, the thermal conductivity of which clearly depends on time and is a power function of temperature with an exponent that depends on time, in the presence of volumetric heat absorption in this medium. The physical properties of the process under study, such as the mode of spatial localization and its variety, stable and metastable localization, are qualitatively considered. The problem of the influence of an instantaneous source on the propagation of a thermal disturbance in an isotropic space is considered. In [17], studies of special boundary value problems for a nonlinear parabolic heat equation were carried out. In the case of a power-law dependence of the thermal



conductivity coefficient on temperature, this equation is used to describe the processes of radiant thermal conductivity, filtration of polytron gas in porous soil, and migration of biological populations. In addition, the equation under consideration has specific nonlinear properties that are interesting from both physical and mathematical points of view. For example, the speed of propagation of the disturbances described by him can be finite. It is shown that in this case the original problem can be reduced to the Cauchy problem for a nonlinear ordinary differential equation of the second order. The work [18] is devoted to the construction of iterative difference schemes for the Dirichlet problem for the Laplace equation. The monograph [19] provides mathematical modeling of the problem of hydrodynamic stability described by nonlinear differential equations using the spectral-grid method. The work [20] illustrates the high accuracy and efficiency of the spectral-grid method when solving the nonlinear equation of stability for single-phase flows. Mathematical modeling and numerical solution of the system of nonlinear stability equations for two-phase flows by the spectral-grid method is described in [21-22]. It is of interest to apply difference iterative schemes to solving a quasilinear heat equation when the heat conductivity coefficient is a nonlinear function of temperature (linear, quadratic, and cubic), as well as to substantiate the effectiveness of the method used by the number of arithmetic operations, and to study the method by the number of arithmetic operations depending on the nonlinearity parameter. The authors are not aware of any work on the solution and study of the above issues.

MAIN PART

Consider a boundary value problem for a quasilinear heat equation with nonlinear heat conductivity coefficients

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + f(u), \quad 0 < x < 1, \quad 0 < t \leq T, \quad (1)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \quad (2)$$

$$u(0, t) = \mu_1(t), \quad u(1, t) = \mu_2(t), \quad 0 \leq t \leq T, \quad (3)$$

Here $k(u) = k_0 u^\sigma$ is the coefficient of thermal conductivity, which is a nonlinear function of temperature, $\sigma \geq 1$.

Continuous area

$$\bar{D} = \{0 \leq x \leq 1, \quad 0 \leq t \leq T\}$$

in which the differential problem (1) - (3) is considered, we introduce the difference grid

$$\bar{\omega}_{h\tau} = \left\{ (x_i, t_j), \quad \begin{array}{l} x_i = ih, \quad i = 0, 1, 2, \dots, N, \quad h = 1/N, \\ t_j = j\tau, \quad j = 0, 1, 2, \dots, M, \quad \tau = T/M \end{array} \right\}.$$

In the difference domain, in accordance with the differential problem (1) - (3), we pose the following difference problems [3]:

scheme a):

$$\begin{aligned} \frac{\hat{y}_i - y}{\tau} &= \frac{1}{h} \left[a_{i+1}(\hat{y}) \frac{\hat{y}_{i+1} - \hat{y}_i}{h} - a_i(\hat{y}) \frac{\hat{y}_i - \hat{y}_{i-1}}{h} \right] + f(\hat{y}_i), & 0 < i < N, \\ & & 0 \leq j < M, \\ y_i^0 &= u_0(x_i), & 0 \leq i \leq N, \\ y_0^{j+1} &= \mu_1(t_{j+1}), \quad y_N^{j+1} = \mu_2(t_{j+1}), & 0 \leq j < M. \end{aligned} \quad (4)$$

scheme b):

$$\begin{aligned} \frac{\hat{y}_i - y}{\tau} &= \frac{1}{h} \left[a_{i+1}(\hat{y}) \frac{\hat{y}_{i+1} - \hat{y}_i}{h} - a_i(\hat{y}) \frac{\hat{y}_i - \hat{y}_{i-1}}{h} \right] + f(\hat{y}_i), & 0 < i < N, \\ & & 0 \leq j < M, \\ y_i^0 &= u_0(x_i), & 0 \leq i \leq N, \\ y_0^{j+1} &= \mu_1(t_{j+1}), \quad y_N^{j+1} = \mu_2(t_{j+1}), & 0 \leq j < M. \end{aligned} \quad (5)$$

In schemes a) and b) $\hat{y}_i = y_i^{j+1}$, $y_i = y_i^j$ and the coefficients $a_i(\mathcal{G}) = a(\mathcal{G}_{i-1}, \mathcal{G}_i)$ are calculated



by one of the following formulas

$$a_i(\mathcal{G}) = 0,5[k(\mathcal{G}_{i-1}) + k(\mathcal{G}_i)],$$

$$a_i(\mathcal{G}) = k\left(\frac{\mathcal{G}_{i-1} + \mathcal{G}_i}{2}\right),$$

$$a_i(\mathcal{G}) = \frac{2k(\mathcal{G}_{i-1})k(\mathcal{G}_i)}{k(\mathcal{G}_{i-1}) + k(\mathcal{G}_i)}.$$

The calculation of the temperature wave strongly depends on how these coefficients are calculated.

The theoretical comparison of the difference schemes a) and b) was carried out in [3] and due to the nonlinearity of the scheme b) the expediency of using the following iterative process is indicated:

$$\frac{y_{i-1}^{(s+1)} - y_i^{(s+1)}}{\tau} = \frac{1}{h} \left[a_{i+1}^{(s)}(y) \frac{y_{i+1}^{(s+1)} - y_i^{(s+1)}}{h} - a_i^{(s)}(y) \frac{y_i^{(s+1)} - y_{i-1}^{(s+1)}}{h} \right] + f(y_i^{(s)}), \quad \begin{matrix} 0 < i < N, \\ 0 \leq s < 3, \\ 0 \leq j < M, \end{matrix}$$

$$y_i^0 = u_0(x_i), \quad 0 \leq i \leq N, \quad (6)$$

$$y_0^{(s+1)} = \mu_1(t_{j+1}^{(s+1)}), \quad y_N^{(s+1)} = \mu_2(t_{j+1}^{(s+1)}), \quad 0 \leq j < M.$$

This scheme is linear with respect to the grid function $y^{(s+1)}$.

To find the value of the grid function $y^{(j+1)}$ from the known values of function y^j , when counting according to scheme (6), you need to make several iterations, and when counting according to scheme a), the value of the grid function $y^{(j+1)}$ on a new layer is found immediately.

Since both schemes are absolutely stable and have the same approximation order $O(\tau + h^2)$, it would seem that in this respect scheme a) has an advantage over the iterative scheme (6). However, practical calculations carried out in this work have shown the high efficiency of the iterative scheme (6).

For this reason, the substantiation of the efficiency of the scheme a) and b) when solving the quasilinear heat equation with nonlinear heat conductivity coefficients with different nonlinearities from the point of view of a computational experiment is of great practical importance. The author is not aware of any work oriented towards solving this problem.

It is known that the main criterion for the effectiveness of any numerical method is the number of arithmetic operations. In this article, the efficiency of the scheme a) and b) are compared by the number of arithmetic operations, when the thermal conductivity coefficient is a nonlinear function of temperature with different nonlinearities, i.e. $k(u) = k_0 u^\sigma$, $\sigma = 1, 2, 3$. The high efficiency of the implicit iteration scheme is illustrated (6).

The results of the performed computational experiments show that with an increase in the value of parameter σ , to obtain the same accuracy according to schemes a) and b), scheme b) allows using a so large time step, which, despite the need to perform iterations, this leads to a decrease in the number of arithmetic operations.

It should be noted that difference schemes (4) and (6) are solved using the sweep method. It is known that the execution of the sweep method on one time layer will require $8N$ arithmetic operations, where N is the number of grid nodes in variable X .

To implement scheme a), $Q_1 = 8N * N1$ arithmetic operations are required, the corresponding number of operations for scheme b) and places like $Q_2 = 8N * IT * N2$, where $8N$ is the number of arithmetic operations in the sweep method, $N1$ and $N2$ are the number of time layers according to schemes a) and b), respectively, IT is the number of iterations according to scheme b) in one time layer.



RESULTS AND DISCUSSION

In the region of

$$\bar{D} = \{0 \leq x \leq 1, 0 \leq t \leq T\}$$

where the differential problem (1) - (3) is considered, we introduce the following difference grid

$$\bar{\omega}_{h\tau} = \left\{ (x_i, t_j), \begin{matrix} x_i = ih, i = 0, 1, 2, \dots, N, h = 1/N, \\ t_j = j\tau, j = 0, 1, 2, \dots, M, \tau = T/M \end{matrix} \right\}$$

To carry out a computational experiment, the parameters of the problem are chosen as follows:

$$N = 50, M = 6, T = 0.6, k(u) = k_0 u^\sigma, \sigma = 1, 2, 3.$$

First, consider case $\sigma = 1$, i.e. $k(u) = k_0 u$ thermal conductivity is a linear function of temperature. The steps of the difference grid were chosen as follows $h = 0.02$ and for the scheme a) $\tau = 0.02$, for the scheme b) $\tau = 0.05$. Computational experiments were carried out according to the schemes a) and b) and the results are shown in Table 1. In the case when $\sigma = 1$ and for the chosen grid steps the number of time layers for scheme a) $N1 = 30$, for scheme b) $N2 = 12$.

Table 1 The results obtained according to schemes a) and b).

$x_i \backslash t_j$		0	0.1	0.2	0.3	0.4	0.5	0.6
		0	0.1	0.2	0.3	0.4	0.5	0.6
i=0	a)	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000
	b)	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000
i=10	a)	0.0400	0.1296	0.3061	0.5654	0.8125	1.0118	1.1712
	b)	0.0400	0.1424	0.3270	0.5596	0.7802	0.9679	1.1253
i=20	a)	0.1600	0.3355	0.5880	0.8468	1.0884	1.2977	1.4714
	b)	0.1600	0.3433	0.5785	0.8208	1.0482	1.2488	1.4205
i=30	a)	0.3600	0.6237	0.8694	1.0847	1.2857	1.4706	1.6331
	b)	0.3600	0.6114	0.8442	1.0545	1.2503	1.4301	1.5909
i=40	a)	0.6400	0.9053	1.0849	1.2436	1.3955	1.5416	1.6777
	b)	0.6400	0.8866	1.0654	1.2239	1.3744	1.5181	1.6530
i=50	a)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000
	b)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000

The difference solutions given in Table 1, obtained according to schemes a) and b), are graphically compared in Figure 1.

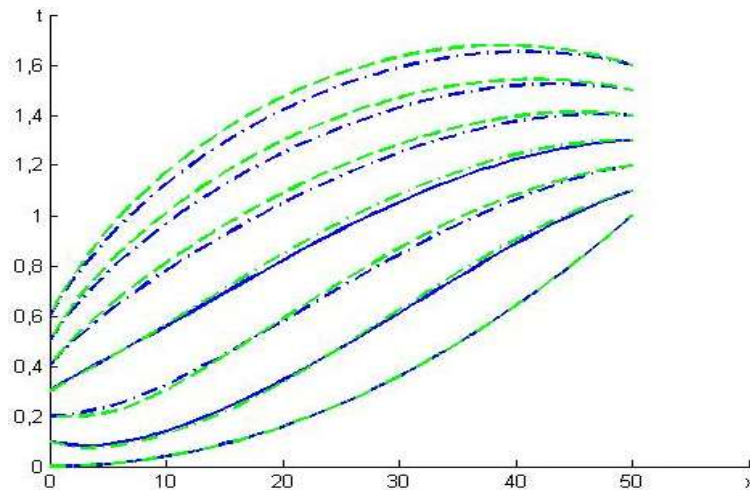


Fig. 1. scheme a) broken line, scheme b) broken line with a point.



From the results of the computational experiment shown in Table 1 and Fig. 1, it can be seen that the difference solutions are somewhat different. To improve the accuracy according to scheme a), we reduce the grid step in time, and the grid step according to scheme b) will remain unchanged, i.e. in scheme a) ($\tau = 0.002, N1 = 300$) and in scheme b) ($\tau = 0.02, N1 = 30$). The results of calculations are shown in Table 2.

Table 2 Results obtained according to schemes a) and b).

t_j x_i		0	0.1	0.2	0.3	0.4	0.5	0.6
		i=0	a) 0	0.1000	0.2000	0.3000	0.4000	0.5000
	b) 0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	
i=10	a) 0.0400	0.1326	0.3238	0.5943	0.8383	1.0309	1.1844	
	b) 0.0400	0.1382	0.3303	0.5828	0.8136	1.0017	1.1552	
i=20	a) 0.1600	0.3443	0.6076	0.8697	1.1121	1.3183	1.4872	
	b) 0.1600	0.3466	0.5973	0.8510	1.0858	1.2879	1.4565	
i=30	a) 0.3600	0.6369	0.8837	1.0994	1.3018	1.4866	1.6465	
	b) 0.3600	0.6268	0.8676	1.0813	1.2807	1.4625	1.6218	
i=40	a) 0.6400	0.9133	1.0918	1.2506	1.4036	1.5504	1.6855	
	b) 0.6400	0.9020	1.0808	1.2396	1.3916	1.5368	1.6714	
i=50	a) 1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	
	b) 1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	

The results obtained in Table 2 by the difference schemes a) and b) are graphically shown in Figures 2.

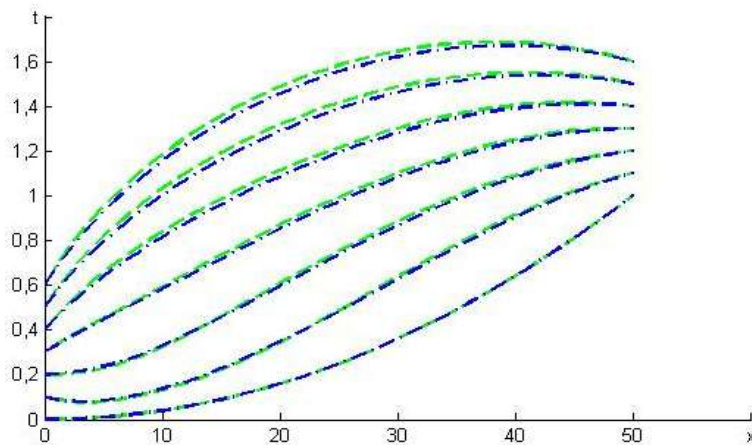


Fig. 2. scheme a) broken line, scheme b) broken line with a point.

The results of the computational experiment show that in order to obtain the accuracy achieved by scheme b) in scheme a) the grid step in time should be reduced by a factor of 10. In this case, the number of arithmetic operations according to scheme a) is equal to $Q_1 = 120,000$, and according to scheme b) is equal to $Q_2 = 36,000$.

In case $\sigma = 2$, i.e. when the thermal conductivity coefficient is a quadratic function of temperature, the steps according to difference schemes a) and b) are chosen the same, i.e. $h = 0.02$ and $\tau = 0.02$. A computational experiment was carried out according to schemes a) and b) and the results obtained are shown in Table 3.

Table 3 Results obtained according to schemes a) and b).

t_j x_i		0	0.1	0.2	0.3	0.4	0.5	0.6
		i=0	a) 0	0.1000	0.2000	0.3000	0.4000	0.5000



	b)	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000
i=10	a)	0.0400	0.1050	0.1942	0.4368	0.8377	1.1150	1.2690
	b)	0.0400	0.1061	0.2164	0.5248	0.8691	1.1001	1.2400
i=20	a)	0.1600	0.2811	0.5564	0.8783	1.1390	1.3553	1.5080
	b)	0.1600	0.3016	0.5949	0.8884	1.1387	1.3424	1.4861
i=30	a)	0.3600	0.6267	0.9111	1.1289	1.3153	1.4845	1.6228
	b)	0.3600	0.6430	0.9108	1.1229	1.3081	1.4741	1.6074
i=40	a)	0.6400	0.9342	1.1132	1.2623	1.4002	1.5323	1.6519
	b)	0.6400	0.9302	1.1075	1.2565	1.3947	1.5261	1.6436
i=50	a)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000
	b)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000

The results obtained according to schemes a) and b) in Table 3 are graphically shown in Figures 3.

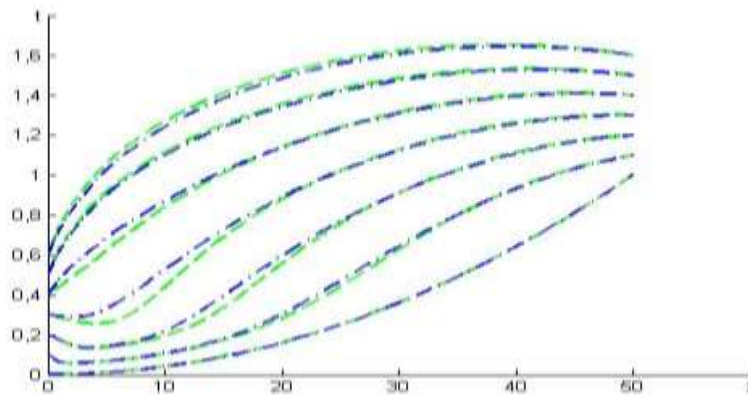


Fig. 3. scheme a) broken line, scheme b) broken line with a point.

From the calculation results given in Table 3 and Figures 3, it can be seen that the difference solutions obtained by schemes a) and b) differ significantly. In order to increase the accuracy according to scheme a), we decrease the grid step in time, and the grid step according to scheme b) will remain unchanged, i.e. in scheme a) ($\tau = 0.0002, N1 = 3000$) and in scheme b) ($\tau = 0.02, N2 = 30$). The obtained numerical results are shown in table 4.

Table 4 Results obtained according to schemes a) and b).

$t_j \backslash x_i$		0	0.1	0.2	0.3	0.4	0.5	0.6
i=0	a)	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000
	b)	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000
i=10	a)	0.0400	0.1067	0.2010	0.5158	0.8939	1.1388	1.2768
	b)	0.0400	0.1061	0.2164	0.5248	0.8691	1.1001	1.2400
i=20	a)	0.1600	0.2882	0.5998	0.9090	1.1630	1.3731	1.5162
	b)	0.1600	0.3016	0.5949	0.8884	1.1387	1.3424	1.4861
i=30	a)	0.3600	0.6514	0.9289	1.1416	1.3265	1.4954	1.6291
	b)	0.3600	0.6430	0.9108	1.1229	1.3081	1.4741	1.6074
i=40	a)	0.6400	0.9433	1.1188	1.2670	1.4047	1.5374	1.6554
	b)	0.6400	0.9302	1.1075	1.2565	1.3947	1.5261	1.6436
i=50	a)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000
	b)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000

Difference solutions obtained by schemes a) and b) and presented in Table 4 are graphically shown in Fig. 4.

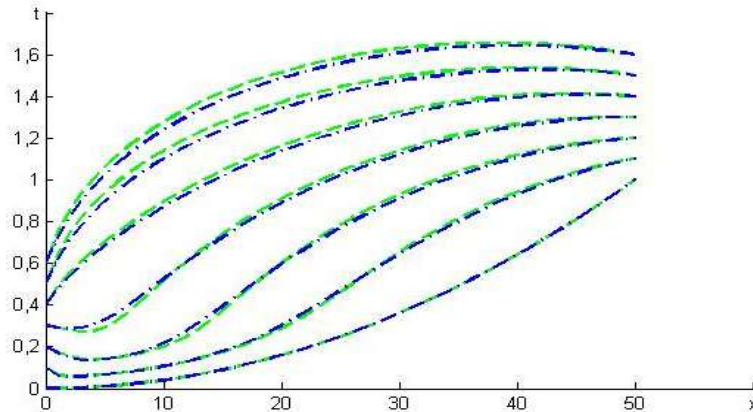


Fig. 4. scheme a) broken line, scheme b) broken line with a point.

The results of the computational experiment carried out in case $\sigma = 2$ show that in order to obtain the accuracy achieved by scheme b) in scheme a) the grid step in time should be reduced by 100 times. In this case, the number of arithmetic operations according to scheme a) is equal to $Q_1 = 1200,000$, and according to scheme b) it is equal to $Q_2 = 36,000$.

In case $\sigma = 3$, i.e. when the thermal conductivity coefficient is a cubic function of temperature, at first the steps of the difference grid will remain unchanged: $h = 0.02$ and $\tau = 0.02$. The obtained numerical results according to schemes a) and b) are shown in Table 5.

Table 5 Results obtained according to schemes a) and b).

t_j x_i		0	0.1	0.2	0.3	0.4	0.5	0.6
		i=0	a) 0	0.1000	0.2000	0.3000	0.4000	0.5000
	b) 0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	
i=10	a)	0.0400	0.1030	0.1808	0.2859	0.8233	1.1849	1.3185
	b)	0.0400	0.1032	0.1818	0.4237	0.8902	1.1611	1.2821
i=20	a)	0.1600	0.2558	0.4564	0.9017	1.1768	1.3825	1.5099
	b)	0.1600	0.2617	0.5651	0.9182	1.1704	1.3644	1.4846
i=30	a)	0.3600	0.6041	0.9388	1.1581	1.3307	1.4824	1.5993
	b)	0.3600	0.6439	0.9400	1.1491	1.3197	1.4692	1.5822
i=40	a)	0.6400	0.9535	1.1304	1.2712	1.3986	1.5195	1.6265
	b)	0.6400	0.9494	1.1234	1.2639	1.3915	1.5121	1.6174
i=50	a)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000
	b)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000

The results shown in Table 5 for schemes a) and b) are graphically shown in Figure 5.

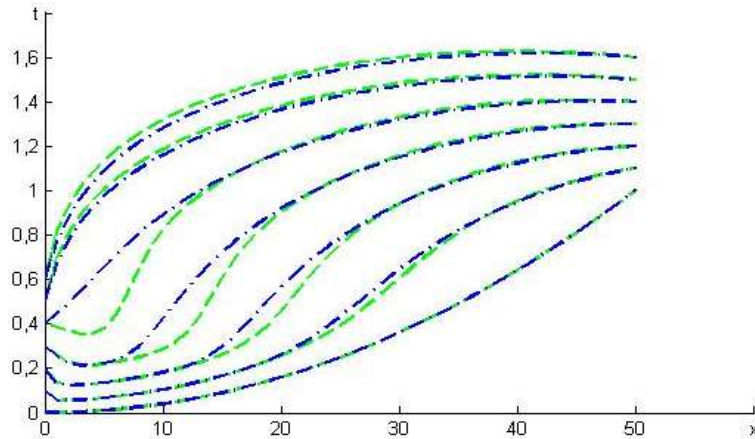


Fig. 5. scheme a) broken line, scheme b) broken line with a point.

From the calculation results given in Table 5 and Figures 5, it can be seen that the difference solutions obtained by schemes a) and b) rarely differ. In order to increase the accuracy according to scheme a), we decrease the time step, and leave the grid step according to scheme b) unchanged, i.e. according to scheme a) ($\tau = 0.00002, N1 = 30000$) and in scheme b) ($\tau = 0.02, N2 = 30$). The results obtained are shown in table 6.

Table 6 Results obtained according to schemes a) and b).

t_j x_i		0	0.1	0.2	0.3	0.4	0.5	0.6
		i=0	a)	0	0.1000	0.2000	0.3000	0.4000
	b)	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000
i=10	a)	0.0400	0.1043	0.1842	0.3254	0.9017	1.1988	1.3182
	b)	0.0400	0.1032	0.1818	0.4237	0.8902	1.1611	1.2821
i=20	a)	0.1600	0.2585	0.5377	0.9356	1.1909	1.3899	1.5100
	b)	0.1600	0.2617	0.5651	0.9182	1.1704	1.3644	1.4846
i=30	a)	0.3600	0.6406	0.9562	1.1656	1.3348	1.4856	1.5992
	b)	0.3600	0.6439	0.9400	1.1491	1.3197	1.4692	1.5822
i=40	a)	0.6400	0.9619	1.1336	1.2728	1.3996	1.5205	1.6263
	b)	0.6400	0.9494	1.1234	1.2639	1.3915	1.5121	1.6174
i=50	a)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000
	b)	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000

The calculation results are shown in Table 6 according to schemes a) and b) are graphically shown in Figure 6.

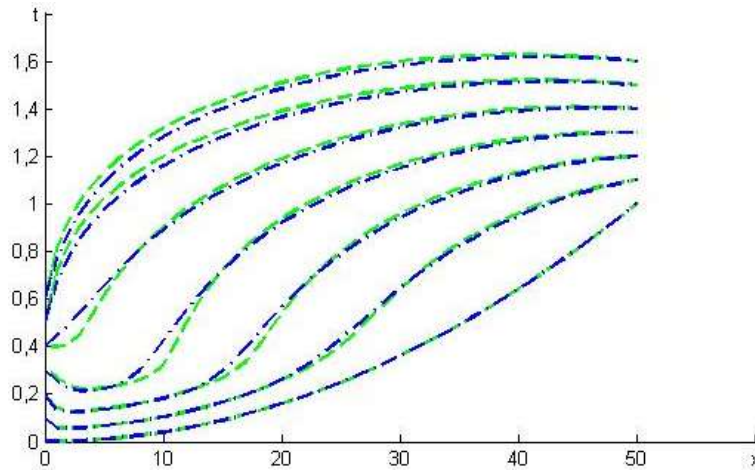


Fig. 6. scheme a) broken line, scheme b) broken line with a point.

The results of the computational experiment carried out in case $\sigma = 3$ show that in order to obtain the accuracy achieved according to scheme b) in scheme a) the grid step in time should be reduced by 1000 times. In this case, the number of arithmetic operations according to scheme a) is equal to $Q_1 = 12,000,000$, and according to scheme b) it is equal to $Q_2 = 36,000$.

Now, the number of arithmetic operations Q spent for different values of the forms of parameter σ will be entered into table 7.

Table 7 Results obtained according to schemes a) and b).

form parameter σ	1	2	3
scheme a)	$12 * 10^4$	$12 * 10^5$	$12 * 10^6$
scheme b)	$36 * 10^3$	$36 * 10^3$	$36 * 10^3$

The results given in Table 7 are graphically depicted in Figure 7. Thus, the nonlinearity of the thermal conductivity coefficient leads to of which the final rate of heat propagation.

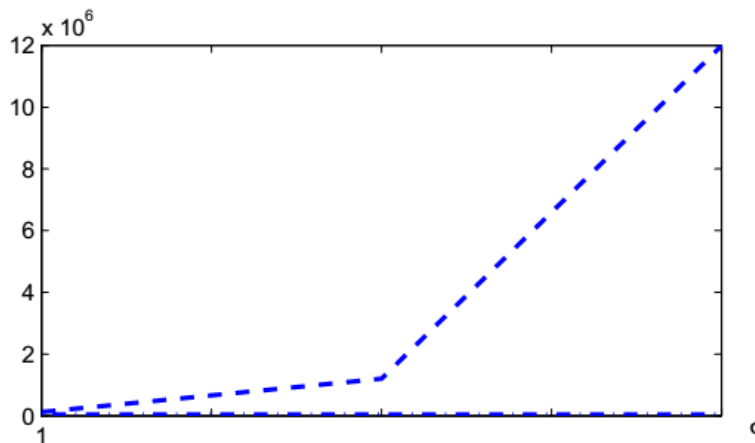


Fig. 7. Scheme A) Broken Line, Scheme B) Broken Line With A Point.

CONCLUSIONS

1. Difference solutions for the quasilinear heat equation are determined, when the heat conductivity coefficient is a nonlinear function of temperature using an implicit and implicit iterative scheme.
2. The implicit and implicit iterative schemes are compared in terms of the number of arithmetic operations, formulas for calculating the numbers of arithmetic operations are derived.
3. The high efficiency of the implicit iterative scheme is shown in solving the formulated differential problem.

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