SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

ON UNCERTAIN MEASURE

[1]Sara Mahdi Abood
University of AL-Muthana, College of sciences, Department of Mathematics

[2]Ahmed Tuaman ALamori
University of AL-Muthana, College of sciences, Department of Mathematics

ABSTRACT
In this work we studied some rustles that related with uncertain measure and this kind of measure is the first basic idea for uncertainty theory which found in (2007) and refined in (2010) by Prof. B. Liu, also we introduced some of the new properties for uncertain measure in addition we introduced the idea of continuous uncertain measure.

INTRODUCTION

Some information and knowledge are usually represented by human language like "about 100km", "approximately 39^{o}C", "roughly 80kg", "low speed", "middle age", and "big size". How do we understand them? Perhaps some people think that they are subjective probability or they fuzzy concepts. However, a lot of surveys showed that those imprecise quantities behaved neither like randomness nor like fuzziness.In other words, those imprecise quantities behaved cannot be quantified by probability measure (Kolmogorove in 1933), capacity (Choquet in 1954), fuzzy measure (Sugeno in 1974), possibility measure (Zadeh in 1978), and credibility measure (B.Liu and Y.Liu in 2002). In order to develop a theory of uncertain measure, (B.Liu) defined a new measure, (B.Liu) founded an uncertainty theory that is a branch of mathematics based on normality, self-duality, countable subadditivity and product measure axioms [9].

This work consists of introduction and two chapters. In chapter one, we recall some basic definitions of sigma field, probability measure uncertain measure, uncertain variable, Also, we recalled first basic some properties of uncertain measure. Chapter two is devoted to continuous uncertain measure and it's Properties.

Chapter 1

In this chapter, we gave some basic definitions, properties, results and some of the basic concepts that related with uncertain measure. In section one, we defined (Sigma field) and recalled some of the applications and theorems that related with it. In sections two, we recalled the definition of the uncertain measure and recalled some of the applications and the first properties for it.

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

§(1.1) Sigma Field

Definition (1.1.1)[1,10]

A family F of subsets of a set Ω is called a σ-field (σ-algebra) on a set Ω,if

(1) $\Omega \in F$

(2) If $A \in F$,then $A^c \in F$

(3) If $A_n \in F, \quad n = 1,2,\cdots$,then $\bigcup_{n=1}^{\infty} A_n \in F$

A measurable Space is a pair (Ω, F), where Ω is a set and F is a σ-field on Ω. A subset A of Ω is called measurable (measurable with respect to the σ-field F), if $A \in F$, i.e., any member of F is called a measurable set.

• It is clear to show that

(1) $\phi \in F$: for $\Omega \in F$ and $\phi = \Omega^c \in F$. (2) If $A_n \in F, \quad n = 1,2,\cdots$,then $\bigcap_{n=1}^{\infty} A_n \in F$, $\limsup_{n\to\infty} A_n \in F$ and $\liminf_{n\to\infty} A_n \in F$. (3) If $A_1, A_2, \ldots, A_n \in F$, then $\bigcup_{i=1}^{n} A_i \in F$ and $\bigcap_{i=1}^{n} A_i \in F$

Example (1.1.2)[1,10]

(1) The family $F = \{\phi, \Omega\}$ is a σ-field on Ω. (2) The family F of all subsets of a set Ω is a σ-field on Ω. (3) The family F of all finite subsets of R is a σ-field on R iff Ω is finite.(4) The family F of all bounded subsets of R is not a σ-field on R.(5) If A is subset of a set of Ω, then $F = \{\phi, A, A^c, \Omega\}$ is a σ-field on Ω.

Theorem (1.1.3)[1]

If $\{F_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary family of σ-field on a set Ω with $\Lambda \neq \phi$, then $F = \bigcap_{\lambda\in\Lambda} F_\lambda$ is a σ-field on Ω .

Definition (1.1.4)[1]

Let G be a family of subsets of a set Ω. The smallest σ-field containing G called the σ-field generated by G and it is denoted by $\sigma(G)$

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

• It is clear to show that(1) $\sigma(G)=$ intersection of all σ-fields on Ω, which contain G.

(2) $\sigma(G)=G$ iff G is σ-field on Ω.

Definition (1.1.5)[1]

Let (Ω,τ) be a topological space. The σ-field generated by τ is called the Borel σ-field and it is denoted by $\beta(\Omega)$, i.e. $\beta(\Omega)=\sigma(\tau)$. The member of $\beta(\Omega)$ are called Borel sets of Ω.

Definition (1.1.6)[1]

Let G be a family of subsets of a set Ω, and let $A\subset\Omega$. The restriction (or trace) of G on A is a collection of all sets by the form $A\cap B$, where $B\in G$, and it is denoted by G_A (or $A\cap G$)

$$G_A = A\cap G = \{A\cap B : B\in G\}$$

G_A is a family of subsets of A. The σ-field $\sigma(G_A)$ generated by G_A some time denoted by $\sigma_A(A\cap G)$, i.e., $\sigma(G_A)=\sigma_A(A\cap G)$

Theorem (1.1.7)[1]

Let G be a family of subsets of a set Ω, and let $A\subset\Omega$

(1) $A\cap\sigma(G)$ is a σ-field on A. (2) $\sigma(G_A)=A\cap\sigma(G)$ (3) If G is closed under finite intersection and $A\in G$, then $G_A=\{B\in G : B\subset A\}$ (4) If G is a σ-field on Ω, then G_A is a σ-field on A.

Definition (1.1.8)[1]

Let F_i be a σ- field of subsets of Ω_i, $i=1,2,\cdots,n$, and let $\Omega=\prod_{i=1}^{n}\Omega_i$. A measurable rectangle in Ω is a set $A=\prod_{i=1}^{n}A_i$, where $A_i\in F_i$ for each $i=1,2,\cdots,n$. The smallest σ-field containing the measurable rectangle is called the product σ- field, and denoted by $\prod_{i=1}^{n}F_i$. If all F_i coincide with fixed σ- field, the product σ- field is denoted by F^n.

§(1.2) Uncertainty Space

Definition (1.2.1)[1]

A measure on a σ-field F is a non-negative, extended real valued function P on F such that whenever $A_1,A_2,\cdots,A_n,\ldots$ form a finite or countably infinite collection of disjoint sets in F, we have $P(\bigcup_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}P(A_n)$ (1.1)

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

- If $P(\Omega)=1$, P is called a probability measure.

- A measure space is a triple (Ω,F,P) where Ω is a set, F is a σ-field on Ω and P is a measure on F. If P is a probability measure, (Ω,F,P) is called probability space.

- If (Ω,F,P) is a probability space, the set Ω is called the sample space, the subsets of Ω which belong to F are called events.

Definition (1.2.2)[7]

Let (Ω,F) be a measurable space. A set function $\mu : F \to R$ is said to be an uncertain measure on F if it satisfies the following axioms:

Axiom 1.(**Normality Axiom**) : $\mu(\Omega)=1$ for the universal set Ω.

Axiom 2.(**Self-Duality Axiom**): $\mu(A)+\mu(A^c)=1$ for any event A.

Axiom 3.(**Countable Subadditivity Axiom**): For every countable sequence of events $\{A_n\}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu(A_n)$$

An uncertainty space is a triple (Ω,F,μ) where Ω is a set, F is a σ-field on Ω, μ is a uncertain measure on F .In 2009, Liu proposed the fourth axiom of uncertainty theory called product measure axiom.

Axiom4.(**Product Measure Axiom**): Let (Ω_k, F_k, μ_k) be uncertainty spaces for $k=1,2,\cdots,n$. Then the product uncertain measure μ is an uncertain measure on the product σ- filed $F_1 \times F_2 \times \cdots \times F_n$ satisfying :

$$\mu\left(\prod_{k=1}^{n} A_k\right) = \min\{\mu_k(A_k) : A_k \in F_k, k=1,2,....n\}$$

That is, for each event $A \in F = F_1 \times F_2 \times \cdots \times F_n$, we have

$$\mu(A)=\begin{cases} \alpha, & \alpha > 0.5 \\ 1-\beta, & \beta > 0.5 \\ 0.5, & o.w \end{cases}$$

where $\begin{array}{l} \alpha = \sup\{\min\{\mu_k(A_k)\} : k=1,2,\cdots,n\} : A_1 \times A_2 \times \cdots \times A_n \subseteq A\}, \\ \beta = \sup\{\min\{\mu_k(A_k)\} : k=1,2,\cdots,n\} : A_1 \times A_2 \times \cdots \times A_n \subseteq A^c\} \end{array}$

Remark (1.2.3)[2]

Probability measure satisfies the above three axioms, probability is not a special case of uncertainty theory because the product probability measure does not satisfy the product measure axiom.

(4) If **Example (1.2.4)[5,7]**

Let $\Omega = \{a,b,c\}$. $F = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \Omega\}$

Define $\mu : F \to R$ by

$$\mu(\phi) = 0, \quad \mu(\{a\}) = 0.6, \quad \mu(\{b\}) = 0.3, \quad \mu(\{c\}) = 0.2,$$
$$\mu(\{a,b\}) = 0.8, \quad \mu(\{a,c\}) = 0.7, \quad \mu(\{b,c\}) = 0.4, \quad \mu(\Omega) = 1$$

Then μ is an uncertain measure because it satisfies the four axioms.

And

(1)If $A_1 = \{a,b\}, \quad A_2 = \{a,c\}$,then $A_1 \cup A_2 = \Omega, \quad A_1 \cap A_2 = \{a\}, \quad A_1 \cap A_2^c = \{b\}$, and

- $\mu(A_1 \cap A_2) + \mu(A_1 \cap A_2^c) = \mu(\{a\}) + \mu(\{b\}) = 0.6 + 0.3 = 0.9, \quad \mu(A_1) = \mu(\{a,b\}) = 0.8$
  $\mu(A_1) \neq \mu(A_1 \cap A_2) + \mu(A_1 \cap A_2^c)$

- $\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) = \mu(\Omega) + \mu(\{a\}) = 1 + 0.6 = 1.6$
  $\mu(A_1) + \mu(A_2) = \mu(\{a,b\}) + \mu(\{a,c\}) = 0.8 + 0.7 = 1.5$
  $\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) \neq \mu(A_1) + \mu(A_2)$

(2) If $A_1 = \{a\}, \quad A_2 = \{b\}$, then $A_1 \cap A_2 = \phi$, and

- $\mu(A_1 \cup A_2) = \mu(\{a,b\}) = 0.8, \quad \mu(\{a\}) + \mu(\{b\}) = 0.6 + 0.3 = 0.9$
  $\mu(A_1 \cup A_2) = \mu(\{a,b\}) \neq \mu(\{a\}) + \mu(\{b\})$

(3) If $A_1 = \{a\}$, $A_2 = \{a,c\}$, then $A_1 \subset A_2$ and $A_2 - A_1 = \{c\}$

- $\mu(A_2 - A_1) = 0.2, \quad \mu(A_2) - \mu(A_1) = 0.7 - 0.6 = 0.1$
  $\mu(A_2 - A_1) \neq \mu(A_2) - \mu(A_1)$

$A_1 = \{a\}, \quad A_2 = \{b\}$, then

- $\mu(A_1 \cap A_2) = \mu(\phi) = 0$
  $\mu(A_1).\mu(A_2) = 0.18$
  $\mu(A_1 \cap A_2) \neq \mu(A_1).\mu(A_2)$

Example (1.2.5)[7]

Suppose that $u : R \to R$ is a nonnegative function satisfying $\sup\{u(x) + u(y) : x \neq y\} = 1$

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

Then for any set A of real numbers, the set function

$$\mu(A)=\begin{cases}\sup\{u(x):x\in A\}, & if \ \sup\{u(x):x\in A\}<0.5\\ 1-\sup\{u(x):x\in A^c\}, & if \ \sup\{u(x):x\in A\}\geq 0.5\end{cases}$$

is an uncertain measure on R.

Ans:

Step1: We prove the normality ,i.e., $\mu(R)=1$.the argument breaks down into two cases .

Case 1: Assume $\sup\{u(x):x\in R\}<0.5$.but this is impossible because $\sup\{u(x)+u(y):x\neq y\}=1$,then $\mu(R)\neq 1$.

Case 2: Assume $\sup\{u(x):x\in R\}\geq 0.5$.then $\mu(R)=1-\sup\{u(x):x\in R^c\}$

$=1-\sup\{\phi\}=1$

Step2:We prove the self-duality ,i.e., $\mu(A)+\mu(A^c)=1$. the argument breaks down into two cases .

Case 1: Assume $\sup\{u(x):x\in A\}<0.5$.

Then $\mu(A)=\sup\{u(x):x\in A\}$

$\mu(A^c)=1-\sup\{u(x):x\in((A^c)^c)\}=1-\mu(A)$.

Case 2: Assume $\sup\{u(x):x\in A\}\geq 0.5$.

Then $\mu(A)=1-\sup\{u(x):x\in A^c\}$

$\mu(A^c)=1-(1-\sup\{u(x):x\in A^c\})=1-\mu(A)$

Step3:We prove the countable subadditivity of μ. For simplicity, we only prove the cases of two events A_1 and A_2. the argument breaks down into two cases .

Case 1: Assume $\mu(A_1)<0.5$ and $\mu(A_2)<0.5$.

Then $\mu(A_1\cup A_2)<0.5$. But since $\sup\{u(x)+u(y):x\neq y\}=1$,

Then $\mu(A_1\cup A_2)\leq\mu(A_1)+\mu(A_2)$.

Case 2: Assume $\mu(A_1)\geq 0.5$ and $M(A_2)<0.5$.When $\mu(A_1\cup A_2)=0.5$, since $\sup\{u(x)+u(y):x\neq y\}=1$

then $\mu(A_1\cup A_2)\leq\mu(A_1)+\mu(A_2)$.

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

Example (1.2.6)[7]

Suppose $u: R \to R$ is a nonnegative and integrable function such that $\int_R u(x)dx \geq 1$.

Then for any Borel set A of real numbers, the set function

$$\mu(A) = \begin{cases} \int_A u(x)dx, if & \int_A u(x)dx < 0.5 \\ 1 - \int_{A^c} u(x)dx, if & \int_{A^c} u(x)dx < 0.5 \\ 0.5, & o.w \end{cases}$$

is an uncertain measure on R.

Ans:

By the same answer of example (1.2.5).

Example (1.2.7)[7]

Suppose $u: R \to R$ is a nonnegative function and $v: R \to R$ is a nonnegative and integrable function such that

$$\sup\{u(x) : x \in A\} + \int_A v(x)dx \geq 0.5 \text{ and |or } \sup\{u(x) : x \in A^c\} + \int_{A^c} v(x)dx \geq 0.5$$

for any Borel set A of real numbers. Then the set function

$$\mu(A) = \begin{cases} \sup\{u(x) : x \in A\} + \int_A v(x)dx, if & \sup\{u(x) : x \in A\} + \int_A v(x)dx < 0.5 \\ 1 - \sup\{u(x) : x \in A^c\} - \int_{A^c} v(x)dx, if & \sup\{u(x) : x \in A^c\} + \int_{A^c} v(x)dx < 0.5 \\ 0.5, & o.w \end{cases}$$

is an uncertain measure on R.

Ans:

By the same answer of example (1.2.5).

The first basic properties of an uncertain measure are collected in the following theorem.

Theorem (1.2.8)[7]

Let (Ω, F, μ) be an uncertainty space, then

(1) $\mu(\phi) = 0$.

(2) If $A_1, A_2 \in F$ and $A_1 \subset A_2$, then $\mu(A_1) \leq \mu(A_2)$

(3) If $A_1, A_2, \cdots, A_n \in F$, then $\mu(\bigcup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} \mu(A_k)$.

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

(4) $\max\{\mu(A_1),\mu(A_2)\} \le \mu(A_1 \cup A_2) \le \mu(A_1)+\mu(A_2)$ for all $A_1, A_2 \in F$.

(5) $\mu(A_1)+\mu(A_2)-1 \le \mu(A_1 \cap A_2) \le \min\{\mu(A_1),\mu(A_2)\}$ for all $A_1, A_2 \in F$.

(6) $0 \le \mu(A) \le 1$ for all $A \in F$.

Proof :

(1)Since $\mu(A)+\mu(A^c)=1$ for all $A \in F$ and $\Omega^c = \phi$, then $\mu(\Omega)+\mu(\phi)=1$

$\Rightarrow \quad \mu(\phi)=1-\mu(\Omega)=1-1=0$

(2)Since $\Rightarrow \quad A_1 \subset A_2 \quad \Rightarrow \quad \Omega = A_1^c \cup A_2 \quad \Rightarrow \quad 1=\mu(\Omega) \le \mu(A_1^c)+\mu(A_2)=1-\mu(A_1)+\mu(A_2)$

$\Rightarrow \quad \mu(A_1) \le \mu(A_2)$.

(3) Put $A_k = \phi$ for all $k \ge n+1$

$\Rightarrow \quad \{A_k\}$ is a sequence of disjoint sets in $F \Rightarrow \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{n} A_k$

$$\mu(\bigcup_{k=1}^{n} A_k) = \mu(\bigcup_{k=1}^{\infty} A_k) \le \sum_{k=1}^{\infty}\mu(A_k) = \sum_{k=1}^{n}\mu(A_k) + \sum_{k=n+1}^{\infty}\mu(A_k) = \sum_{k=1}^{n}\mu(A_k)+\mu(\phi) = \sum_{k=1}^{n}\mu(A_k)$$

(4) Let $A_1, A_2 \in F$

since $A_1 \subseteq A_1 \cup A_2, \quad A_2 \subseteq A_1 \cup A_2 \quad \Rightarrow \quad \mu(A_1) \le \mu(A_1 \cup A_2), \quad \mu(A_2) \le \mu(A_1 \cup A_2)$

$\max\{\mu(A_1),\mu(A_2)\} \le \mu(A_1 \cup A_2)$

since from (3) , we have $\mu(A_1 \cup A_2) \le \mu(A_1)+\mu(A_2)$

hence $\max\{\mu(A_1),\mu(A_2)\} \le \mu(A_1 \cup A_2) \le \mu(A_1)+\mu(A_2)$ for all $A_1, A_2 \in F$

(5) Let $A_1, A_2 \in F$

(a) Since $\mu(A_1^c \cup A_2^c) \le \mu(A_1^c)+\mu(A_2^c)$

$\Rightarrow \quad -\mu(A_1^c \cup A_2^c) \ge -\mu(A_1^c)-\mu(A_2^c) = -(1-\mu(A_1))-(1-\mu(A_2)) = \mu(A_1)+\mu(A_2)-2$

$$\mu(A_1 \cap A_2) = 1-\mu((A_1 \cap A_2)^c) = 1-\mu(A_1^c \cup A_2^c) \ge 1+\mu(A_1)+\mu(A_2)-2 = \mu(A_1)+\mu(A_2)-1$$

Hence, $\mu(A_1)+\mu(A_2)-1 \le \mu(A_1 \cap A_2)$ for all $A_1, A_2 \in F$

(b)Since $A_1 \cap A_2 \subseteq A_1, \quad A_1 \cap A_2 \subseteq A_2 \quad \Rightarrow \quad \mu(A_1 \cap A_2) \le \mu(A_1), \quad \mu(A_1 \cap A_2) \le \mu(A_2)$

$\Rightarrow \quad \mu(A_1 \cap A_2) \le \min\{\mu(A_1),\mu(A_2)\}$

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

Therefore $\mu(A_1)+\mu(A_2)-1 \le \mu(A_1 \cap A_2) \le \min\{\mu(A_1), \mu(A_2)\}$ for all $A_1, A_2 \in F$

(6) Let $A \in F$

Since $\phi \subset A \subset \Omega$, then $\mu(\phi) \le \mu(A) \le \mu(\Omega)$

Since $\mu(\phi)=0$, $\mu(\Omega)=1$, then $0 \le \mu(A) \le 1$.

Chapter2

In this chapter we have reviewed Null-Additivity theorem and Asymptotic theorem . Also, we have defined continuous uncertain measure and recalled some properties that related with it.

§(2.1) Sequences of Sets

Let $\{x_n\}$ be a sequence of real numbers. Define

$\limsup_{n\to\infty} x_n = \inf\{\sup\{x_m : m \ge n\} : n \ge 1\}$ and $\liminf_{n\to\infty} x_n = \sup\{\inf\{x_m : m \ge n\} : n \ge 1\}$

If $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$, we say that the limit exists and write $\lim_{n\to\infty} x_n$

Definition(2.1.1)[1]

Let $\{A_n\}$ be a sequence of subsets of a set Ω. The set of all points which belong to infinitely many sets of the sequence $\{A_n\}$ is called the upper limit (or limit superior) of $\{A_n\}$ and (in symbol A^*) and defined by

$$A^* = \limsup_{n\to\infty} {}_n A_n = \{x \in A_n : \text{ for infinitely many } n\} = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \lim_{n\to\infty}\bigcup_{k=n}^{\infty} A_k$$

Thus,

$x \in A^*$ iff for all n, then $x \in A_k$ for some $k \ge n$

The lower limit (or limit inferior) of $\{A_n\}$, denoted by A_* is the set of all points which belong to almost all sets of the sequence $\{A_n\}$, and defined by $A_* = \liminf_{n\to\infty} {}_n A_n = \{x \in A_n :$ for all but finitely many $n\} = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \lim_{n\to\infty}\bigcap_{k=n}^{\infty} A_k$

Thus,

$x \in A_*$ iff for some n, then $x \in A_k$ for all $k \ge n$

Theorem (2.1.2)[1]

Let $\{A_n\}$ be a sequence of subsets of a set Ω.

(1) $(\lim_{n\to\infty}\sup_n A_n)^c = \lim_{n\to n}\inf_n A_n^c$ (2) $(\lim_{n\to\infty}\inf_n A_n)^c = \lim_{n\to\infty}\sup_n A_n^c$

(3) $\lim_{n\to\infty}\inf_n A_n \subseteq \lim_{n\to\infty}\sup_n A_n$

Proof:

Since $\lim_{n\to\infty}\sup_n A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$

$$(\lim_{n\to\infty}\sup_n A_n)^c = (\bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} A_k))^c = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} A_k^c) = \lim_{n\to\infty}\inf_n A_n^c .$$

(2)Since $\lim_{n\to\infty}\inf_n A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k$

$$(\lim_{n\to\infty}\inf_n A_n)^c = (\bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} A_k))^c = \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} A_k^c) = \lim_{n\to\infty}\sup_n A_n^c .$$

(3)Since $\bigcap_{k=n}^{\infty} A_k \subseteq \bigcup_{k=n}^{\infty} A_k \Rightarrow \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k \subseteq \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \Rightarrow \lim_{n\to\infty}\inf_n A_n \subseteq \lim_{n\to\infty}\sup_n A_n$.

Definition (2.1.3)[1]

A sequence $\{A_n\}$of subsets of a set Ω is said to converge if $\lim_{n\to\infty}\sup_n A_n = \lim_{n\to\infty}\inf_n A_n = A$ (say)

And A is said to be the limit of $\{A_n\}$, we write $A = \lim_{n\to\infty} A_n$ or $A_n \to A$

Definition (2.1.4)[1]

A sequence $\{A_n\}$of subsets of a set Ω is said to be increasing if $A_n \subset A_{n+1}$ for $n = 1,2,\cdots$. It is said to be decreasing if $A_{n+1} \subset A_n$ for $n = 1,2,\cdots$. A monotone sequence of sets is one which either increasing or decreasing.

Theorem(2.1.5)[1]

Any monotone sequence is converge. But the converse is not true.

Proof:

Let $\{A_n\}$ be a monotone sequence of subsets of a set Ω.

If $\{A_n\}$ is an increasing $\Rightarrow$ $A_n \subset A_{n+1}$ for $n = 1,2,\cdots$

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

$\Rightarrow \quad \bigcup_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{k=n}^{\infty} A_k = A_n$ for $n = 1,2,\cdots$

$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} A_k) = \bigcap_{n=1}^{\infty}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} A_n$ and $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} A_k) = \bigcup_{n=1}^{\infty} A_n$ Thus,

$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$

Therefore the sequence $\{A_n\}$ is converge, while, if $\{A_n\}$ is decreasing

$\Rightarrow \quad A_{n+1} \subset A_n$ for $n = 1,2,\cdots \Rightarrow \quad \bigcup_{k=n}^{\infty} A_k = A_n$ and $\bigcup_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n$ for $n = 1,2,\cdots$

$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} A_k) = \bigcap_{n=1}^{\infty}(\bigcup_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} A_n$ and $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} A_k) = \bigcap_{n=1}^{\infty} A_n$

Thus $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$ **.** Therefore the sequence $\{A_n\}$ is converge.

<u>Example (2.1.6)[1]</u>

Let $\Omega = R$ and $A_n = \begin{cases} (0,1-\frac{1}{n}] \ , \ n \quad odd \\ [\frac{1}{n},1) \quad , n \quad even \end{cases}$, then the $\{A_n\}$ is converge but not monotone.

<u>Remark (2.1.7)[1]</u>

If $\{A_n\}$ is an increasing sequence of subsets of a set Ω and $\bigcup_{n=1}^{\infty} A_n = A$, we say that the A_n from an increasing sequence of a set with limit A, or that the A_n increase to A, we write $A_n \uparrow A$. Also If $\{A_n\}$ is a decreasing sequence of subsets of a set Ω and $\bigcap_{n=1}^{\infty} A_n = A$, we say that the A_n from a decreasing sequence of a set with limit A, or that the A_n decrease to A, we write $A_n \downarrow A$.

<u>Theorem (2.1.8)[1]</u>

Let $\{A_n\}$ be a sequence of subsets of a set Ω and let $A \subset \Omega$

(1) If $A_n \uparrow A$, then $A_n^c \downarrow A^c$

(2) If $A_n \downarrow A$, then $A_n^c \uparrow A^c$

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

Proof :

(1) Since $A_n \uparrow A \Rightarrow A_n \subset A_{n+1}$ for $n = 1,2,\cdots$ and $\bigcup_{n=1}^{\infty} A_n = A$

$\Rightarrow A_{n+1}^c \subset A_n^c$ for $n = 1,2,\cdots$ and $\bigcap_{n=1}^{\infty} A_n^c = (\bigcup_{n=1}^{\infty} A_n)^c = A^c \Rightarrow A_n^c \downarrow A^c$.

(2) Since $A_n \downarrow A \Rightarrow A_{n+1} \subset A_n$ for $n = 1,2,\cdots$ and $\bigcap_{n=1}^{\infty} A_n = A$

$\Rightarrow A_n^c \subset A_{n+1}^c$ for $n = 1,2,...$ and $\bigcup_{n=1}^{\infty} A_n^c = (\bigcap_{n=1}^{\infty} A_n)^c = A^c \Rightarrow A_n^c \uparrow A^c$.

<u>Theorem (2.1.9) (Null-Additivity Theorem)[7]</u>

Let (Ω, F, μ) be an uncertainty space, and let $\{A_n\}$ be a sequence of events in F with $\mu(A_n) \to 0$ as $n \to \infty$. Then for any $A \in F$, we have $\lim_{n\to\infty} \mu(A \cup A_n) = \lim_{n\to\infty} \mu(A - A_n) = \mu(A)$

Proof :

Since $A \subset A \cup A_n \Rightarrow \mu(A) \le \mu(A \cup A_n)$ for each n, and since $\mu(A \cup A_n) \le \mu(A) + \mu(A_n)$ for each n. It follows that

$$\mu(A) \le \mu(A \cup A_n) \le \mu(A) + \mu(A_n)$$

For each n. Thus we get $\mu(A \cup A_n) \to \mu(A)$ by using $\mu(A_n) \to 0$.

Since $(A - A_n) \subset A \subset ((A - A_n) \cup A_n)$, we have

$\mu(A - A_n) \le \mu(A) \le \mu(A - A_n) + \mu(A_n)$.

Hence $\mu(A - A_n) \to \mu(A)$ by using $\mu(A_n) \to 0$.

<u>Remark (2.1.10)[7]</u>

It follows from the above theorem that the uncertain measure is null-additive, i.e., $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ if either $\mu(A_1) = 0$ or $\mu(A_2) = 0$. In other words, the uncertain measure remains unchanged if the event is enlarged or reduced by an event with uncertain measure zero.

<u>Theorem (2.1.11) (Asymptotic Theorem)[7]</u>

Let (Ω, F, μ) be an uncertainty space. For any sequence $\{A_n\}$ of events in F, we have

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

$\lim_{n\to\infty} \mu(A_n) > 0$ if $A_n \uparrow \Omega$ and $\lim_{n\to\infty} \mu(A_n) < 1$ if $A_n \downarrow \phi$

Proof :

Assume $A_n \uparrow \Omega$. Since $\Omega = \bigcup_{n=1}^{\infty} A_n$, it follows from the countable subadditivity axiom that

$1 = \mu(\Omega) \le \sum_{n=1}^{\infty} \mu(A_n)$.

Since $\mu(A_n)$ is increasing with respect to n, we have $\lim_{n\to\infty} \mu(A_n) > 0$. If $A_n \downarrow \phi$, then $A_n^c \uparrow \Omega$. It follows from the first inequality and self-duality axiom that $\lim_{n\to\infty} \mu(A_n) = 1 - \lim_{n\to\infty} \mu(A_n^c) < 1$

which complete the proof.

<u>Example (2.1.12)[7]</u>

Assume Ω is the set of real numbers. Let α be a number with $0 < \alpha < 0.5$. Define a set function as follows,

$$\mu(A) = \begin{cases} 0, & A = \phi \\ \alpha, & A \text{ is upper bounded} \\ 0.5, & A \text{ and } A^c \text{ are upper unbounded} \\ 1-\alpha, & A^c \text{ is upper bounded} \\ 1, & A = \Omega \end{cases}$$

It is easy to verify that μ is an uncertain measure. Write $A_n = (-\infty, n]$ for $n = 1,2,\cdots$.

Then $A_n \uparrow \Omega$ and $\lim_{n\to\infty} \mu(A_n) = \alpha$. Furthermore, we have $A_n^c \downarrow \phi$ and $\lim_{n\to\infty} \mu(A_n^c) = 1-\alpha$.

<u>Theorem (2.1.13)[2]</u>

Let (Ω, F, μ) be an uncertainty space. For any sequence $\{A_n\}$ of events in F , we have

(1)If $A_n \uparrow A$, then $\lim_{n\to\infty} \mu(A_n) \le \mu(A)$.

(2)If $A_n \downarrow A$, then $\lim_{n\to\infty} \mu(A_n) \ge \mu(A)$.

Proof :

(1)Since $A_n \uparrow A \Rightarrow A_n \subseteq A_{n+1}$ for $n = 1,2,\ldots$, and $\bigcup_{n=1}^{\infty} A_n = A \Rightarrow A_n \subset A$ for $n = 1,2,\ldots$
$\Rightarrow \mu(A_n) \le \mu(A)$ for $n = 1,2,\ldots$

SJIF Impact Factor: 7.001| ISI I.F.Value:1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 5 | Issue: 12 | December 2020 - Peer Reviewed Journal

so that $\lim_{n\to\infty}\mu(A_n)\le\mu(A)$.

(2)Since $A_n \downarrow A \Rightarrow A_n^c \uparrow A^c \Rightarrow \lim_{n\to\infty}\mu(A_n^c)\le\mu(A^c) \Rightarrow \lim_{n\to\infty}(1-\mu(A_n))\le 1-\mu(A)$

$\Rightarrow \lim_{n\to\infty}\mu(A_n)\ge\mu(A)$.

§(2.2) Continuous Uncertain measure and it's properties

Definition (2.2.1)[3]

Let (Ω, F, μ) be an uncertainty space. We say that μ is

(1) continuous from above at $A\in F$, if $\lim_{n\to\infty}\mu(A_n)=\mu(A)$ whenever $\{A_n\}$ of events in F with $A_n \downarrow A$.

(2) continuous from below at $A\in F$, if $\lim_{n\to\infty}\mu(A_n)=\mu(A)$ whenever $\{A_n\}$ of events in F with $A_n \uparrow A$.

μ is called continuous from above if it is continuous from above at A for all $A\in F$, also μ is called continuous from below at A for all $A\in F$.

Theorem (2.2.2)[2]

Let (Ω, F, μ) be an uncertainty space. Then the following statements are equivalent:

(1) μ is continuous from above at $A\in F$.(2) μ is continuous from below at $A\in F$.

Proof:

(1)$\Rightarrow$(2)

Let $\{A_n\}$ be a sequence of events in F with $A_n \uparrow A$, we have $A_n^c - A^c \downarrow \phi \Rightarrow \lim_{n\to\infty}\mu(A_n^c - A^c)=\mu(\phi)=0$.

Since

$$\mu(A_n^c - A^c)\ge\mu(A_n^c)-\mu(A^c) \Rightarrow \lim_{n\to\infty}\mu(A_n^c - A^c)\ge\lim_{n\to\infty}\mu(A_n^c)-\mu(A^c)$$

$$\Rightarrow \lim_{n\to\infty}\mu(A_n^c)-\mu(A^c)\le 0 \Rightarrow 1-\lim_{n\to\infty}\mu(A_n)-1+\mu(A)\le 0 \Rightarrow \lim_{n\to\infty}\mu(A_n)\ge\mu(A)$$

Since

$\lim_{n\to\infty}\mu(A_n)\le\mu(A) \Rightarrow \lim_{n\to\infty}\mu(A_n)=\mu(A)$. That is μ is continuous from below at $A\in F$.

(2)$\Rightarrow$(1)

Let $\{A_n\}$ be a sequence of events in F with $A_n \downarrow A$, we have $A_n^c \cup A \uparrow \Omega \Rightarrow \lim_{n\to\infty} \mu(A_n^c \cup A) = \mu(\Omega) = 1$.

Since $\mu(A_n^c \cup A) \leq \mu(A_n^c) + \mu(A) \Rightarrow \lim_{n\to\infty} \mu(A_n^c \cup A) \leq \lim_{n\to\infty} \mu(A_n^c) + \mu(A)$

$1 \leq \lim_{n\to\infty} \mu(A_n^c) + \mu(A) \Rightarrow 1 \leq 1 - \lim_{n\to\infty} \mu(A_n) + \mu(A) \Rightarrow \lim_{n\to\infty} \mu(A_n) \leq \mu(A)$

Since $\lim_{n\to\infty} \mu(A_n) \geq \mu(A) \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A)$. That is μ is continuous from above at $A \in F$.

Theorem (2.2.3)[3]

Let (Ω, F, μ) be an uncertainty space. For any sequence $\{A_n\}$ of events in F .

(1) If μ is continuous from above, then $\limsup_{n\to\infty} \mu(A_n) \leq \mu(\limsup_{n\to\infty} A_n)$.

(2) If μ is continuous from below, then $\liminf_{n\to\infty} \mu(A_n) \geq \mu(\liminf_{n\to\infty} A_n)$.

Proof:

Since $\bigcup_{n=k}^{\infty} A_n$ is an increasing sequence and $\bigcup_{n=k}^{\infty} A_n \supset A_k$, we get

$$\mu(\limsup_{n\to\infty} A_n) = \mu(\lim_{n\to\infty} \bigcup_{n=k}^{\infty} A_n) = \lim_{n\to\infty} \mu(\bigcup_{n=k}^{\infty} A_n) \geq \limsup_{n\leftarrow\infty} \mu(A_n)$$

(2) Similarly $\bigcap_{n=k}^{\infty} A_n$ is decreasing and $\bigcap_{n=k}^{\infty} A_n \subset A_n$. Thus

$$\mu(\liminf_{n\to\infty} A_n) = \mu(\lim_{n\to\infty} \bigcap_{n=k}^{\infty} A_n) = \lim_{n\to\infty} \mu(\bigcap_{n=k}^{\infty} A_n) \leq \liminf_{n\to\infty} \mu(A_n).$$

Definition (2.2.4)[3,7]

Let (Ω, F, μ) be an uncertainty space. μ is called continuous if for any sequence $\{A_n\}$ of events in F with $\lim_{n\to\infty} A_n$ exists, we have $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$. The triple (Ω, F, μ) is called a continuous uncertainty space if μ is continuous.

Theorem (2.2.5)[2]

Let (Ω, F, μ) be an uncertainty space. Then μ is continuous if and only if it is continuous from above (or continuous from below) .

Proof: Suppose μ is continuous from below, then μ is continuous from above theorem (2.2.3), we have

$$\mu(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} \mu(A_n) \le \limsup_{n\to\infty} \mu(A_n) \le \mu(\limsup_{n\to\infty} A_n)$$

Since $\lim_{n\to\infty} A_n$ exists, we get the equation $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$.

Example (2.2.6)[11]

Let $f, g : R \to R$ be two nonnegative functions such that $\int_R f(x)dx = 1$, $\inf\{g(x) : x \in R\} = 0$ and $g(x) \le 1$ for any $x \in R$. For any Borel set A of real numbers, define a set function μ as follows:

$\mu(A) = \frac{1}{2}\int_A f(x)dx + \frac{1}{4}(\inf\{g(x) : x \in A^c\} + 1 - \inf\{g(x) : x \in A\})$ Then μ is a continuous uncertain measure.

REFERENCES

1. *R.B. Ash." Real analysis and probability ", Academic process, New York, 1992.*
2. *AL-Zubaidy Zainab H. A. "Some problems related with uncertain me-asure" College of Mathematics and Computer Science , Al- Qadissiya University*
3. *X. Gao,"Some properties of continuous uncertain measure",International Journal of Uncertainty, Fuzziness and Knowledge-Based System , Vol.17,No.3,PP. 419-426, 2009.*
4. *B. Liu,"Uncertainty theory ", 2 nd ed ., Springer-Verlag, Berlin, 2007.*
5. *"Theory and practice of uncertain programming", 3ed., UTLAB , 2009.*
6. *" Uncertainty theory: A Branch of Mathematics for Modeling Human Uncertainty, Springer-Verlag, Berlin, 2010.*
7. *"Uncertainty theory", 4th ed ., UTLAB, 2011.*
8. *"Some research problems in uncertainty theory", Journal of Uncertain Systems, Vol.3, No.1, PP.3-10, 2009.*
9. *"Uncertain set theory and uncertain inference rule with application to uncertain control", Journal of Uncertain Systems, Vol.4, No.2, PP.83-98, 2010.*
10. *D. Williams,"Probablity with martingale", Cambridge University Press, 1991.*
11. *[11] Z. Zhang,"Some discussions on uncertain measure", Fuzzy Optimization and Decision Making, DOI 10.1007/s10700-010-9091-0, 2011.*