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THE KNICKKNACK OF A GENERALIZED FORMULA

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ABSTRACT

In this paper, we have developed the generalized expression of

$${}_2F_1\left[\begin{array}{lll} a, & -n-a\,; & \frac{1}{2} \\ c & ; & \frac{1}{2} \end{array}\right]$$

and it's corresponding integral form and some amazing results from the Generalized formula.

KEY WORDS: *Appell's Hypergeometric Function, Pochhammer symbol.*
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1. Introduction

Appell's Hypergeometric Function $F_1(x,y), F_2(x,y), F_3(x,y), F_4(x,y)$ are defined as(see[7]):

$$F_1[a;b,c;d;x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n\, x^m\, y^n}{(d)_{m+n}\, m!\, n!}, \qquad \max\{|x|,|y|\} < 1 \tag{1.1}$$

$$F_2[a;b,c;d,e;x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n\, x^m\, y^n}{(d)_m(e)_n\, m!\, n!}, \qquad |x|+|y| < 1 \tag{1.2}$$

$$F_3[a,b;c,d;e;x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(b)_n(c)_m(d)_n\, x^m\, y^n}{(e)_{m+n}\, m!\, n!}, \qquad \max\{|x|,|y|\} < 1 \tag{1.3}$$

$$F_4[a,b;c,d;x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}\, x^m\, y^n}{(c)_m(d)_n\, m!\, n!}, \qquad \sqrt{|x|}+\sqrt{|y|} < 1 \tag{1.4}$$

A generalized hypergeometric function ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(k+b_2)...(k+b_q)(k+1)}\, z. \tag{1.5}$$

Where $k+1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{ccc} a_1,a_2,\cdots,a_p & ; & \\ & & z \\ b_1,b_2,\cdots,b_q & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k z^k}{(b_1)_k(b_2)_k\cdots(b_q)_k k!} \tag{1.6}$$

where the parameters $b_1,b_2,\cdots,b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \le q$, converges for $|z| < 1$ if $p = q+1$, diverges for all z , $z \neq 0$ if $p > q+1$[Luke p.156(3)].

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The function ${}_2F_1(a,b;c;z)$ corresponding to $p=2, q=1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0 \tag{1.7}$$

The solution of this equation is

$$y = A_0\Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] \tag{1.8}$$

This is the so-called regular solution, denoted

$${}_2F_1(a,b;c;z) = \Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] = \sum_{k=0}^{\infty}\frac{(a)_k\ (b)_k z^k}{(c)_k k!} \tag{1.9}$$

which converges if c is not a negative integer for all $|\ z\ |<1$ and on the unit circle $|\ z\ |=1$ if $R(c-a-b)>0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.
Kummer's function of the first kind M is a generalized hypergeometric series introduced in (Kummer 1837), given by:

$$M(a,b,z) = \sum_{n=0}^{\infty}\frac{(a)_n\ z^n}{(b)_n\ n!} = {}_1F_1(a;b;z) \tag{1.10}$$

The Jacobi polynomials are defined as follows:[1]

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\Big(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1}{2}(1-z)\Big) \tag{1.11}$$

Pochhammer symbol:

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial,falling sequential product, or lower factorial) is defined as the polynomial[Steffensen p.8]

$$(x)_n = x(x-1)(x-2).......(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k) \tag{1.12}$$

2. Main Generalized Formula

$${}_2F_1\left[\begin{matrix} a, & -n-a\ ; & \frac{1}{2} \\ c & ; & \end{matrix}\right] =$$

$$= \sum_{k=0}^{\infty}\Big[\frac{\Big\{2^{-c-k-n}\ \Gamma(c)\ \Gamma(-c-n)(-a+c)_k\ (a+c+n)_k\Big\}}{k!\ \Gamma(a)\ \Gamma(-a-n)(1+c+n)_k} + \frac{\Big\{2^{-k}\ \Gamma(c)\ \Gamma(c+n)(a)_k\ (-a-n)_k\Big\}}{k!\ \Gamma(-a+c)\ \Gamma(a+c+n)(1-c-n)_k}\Big] \tag{2.1}$$

for $c+n \notin z$

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3.Integral Representation of Main Generalized Formula

$$ {}_2F_1\left[\begin{matrix} a, & -n-a\,; & \frac{1}{2} \\ c & ; & \end{matrix}\right] = \frac{1}{\Gamma(-a-n)} \int_0^\infty e^{-t}\, t^{-1-a-n}\; {}_1F_1[a;c;\frac{t}{2}]\, dt \text{ for } Re(a+n)<0. \quad (3.1) $$

$$ = \frac{\Gamma(c)}{\Gamma(a+c+n)\Gamma(-a-n)} \int_0^1 (1-t)^{-1+a+c+n}\left(1-\frac{t}{2}\right)^{-a} t^{-1-a-n}\, dt \text{ for } Re(c) > -Re(a+n) > 0. \quad (3.2) $$

$$ = \frac{\Gamma(c)}{\Gamma(a+c+n)\Gamma(-a-n)} \int_0^\infty (1+t)^{a-c}\left(t+\frac{1}{2}\right)^{-a} t^{-1+a+c+n}\, dt \text{ for } Re(c) > -Re(a+n) > 0. \quad (3.3) $$

$$ = -\frac{\iota\,\Gamma(c)}{2\pi\,\Gamma(a)\,\Gamma(-a+c)\,\Gamma(a+c+n)\,\Gamma(-a-n)} \int_{-\iota\infty+\gamma}^{\iota\infty+\gamma} 2^s\,\Gamma(a-s)\,\Gamma(-a-n-s)\,\Gamma(s)\,\Gamma(c+n+s)ds \quad (3.4) $$

$$ \text{for } max(0,-Re(c+n)) < \gamma < min(Re(a),-Re(a+n)) $$

4. Special cases of the formula

If $a=c=0$, then

$$ {}_2F_1\left[\begin{matrix} 0, & -n\,; & \frac{1}{2} \\ 0 & ; & \end{matrix}\right] = \left(\frac{\Gamma(0)\,\Gamma(1)\,P_0^{(-1,-n)}(0)}{\Gamma(0)} = \Gamma(1)\,P_0^{(-1,-n)}(0)\right). \quad (4.1) $$

$$ = \sum_{k=0}^{\infty} \frac{{}_2F_1(k,k-n;k;z_0)(-n)_k(\frac{1}{2}-z_0)^k}{k!} \quad \text{for(not}(z_0 \in R \text{ and } 1 \le z_0 \le \infty)) \quad (4.2) $$

$$ = \frac{1}{\Gamma(-n)} \int_0^\infty e^{-t}\, t^{-1-n}{}_1F_1\left(0;0;\frac{t}{2}\right) dt \quad \text{for } Re(n) < 0 \quad (4.3) $$

If $a=c=1$, then

$$ {}_2F_1\left[\begin{matrix} 1, & -n-1\,; & \frac{1}{2} \\ 1 & ; & \end{matrix}\right] = \Gamma(1)\,P_{-1}^{(0,-1-n)}(0) \quad (4.4) $$

$$ = \sum_{k=0}^{\infty} \frac{{}_2F_1(1+k,-1+k-n;1+k;z_0)(-1-n)_k(\frac{1}{2}-z_0)^k}{k!} \quad \text{for(not}(z_0 \in R \text{ and } 1 \le z_0 \le \infty)) \quad (4.5) $$

$$ = \frac{\sum_{j=0}^{\infty} Res_{s=-j}(-\frac{1}{2})^{-s}\,\Gamma(-1-n-s)\,\Gamma(s)}{\Gamma(-1-n)} \quad (4.6) $$

$$ = -\frac{2}{\Gamma(2+n)\,\Gamma(-1-n)} \int_0^1 \frac{(1-t)^{1+n}\,t^{-2-n}}{-2+t}\, dt \quad \text{for } -2 < Re(n) < -1 \quad (4.7) $$

$$ = \frac{1}{\Gamma(2+n)\,\Gamma(-1-n)} \int_0^\infty \frac{t^{1+n}}{\frac{1}{2}+t}\, dt \quad \text{for } -2 < Re(n) < -1 \quad (4.8) $$

$$ = \frac{1}{\Gamma(-1-n)} \int_0^\infty e^{-t}\, t^{-2-n}{}_1F_1\left(1;1;\frac{t}{2}\right) dt \quad \text{for } 1+Re(n) < 0 \quad (4.9) $$

If $a=c=2$, then

$$ {}_2F_1\left[\begin{matrix} 2, & -n-2\,; & \frac{1}{2} \\ 2 & ; & \end{matrix}\right] = \left(\frac{\Gamma(-1)\,\Gamma(2)\,P_{-2}^{(1,-2-n)}(0)}{\Gamma(0)}\right). \quad (4.10) $$

$$ = \sum_{k=0}^{\infty} \frac{{}_2F_1(2+k,-2+k-n;2+k;z_0)(-2-n)_k(\frac{1}{2}-z_0)^k}{k!} \quad \text{for(not}(z_0 \in R \text{ and } 1 \le z_0 \le \infty)) \quad (4.11) $$

$$= \frac{\sum_{j=0}^{\infty} Res_{s=-j}(-\frac{1}{2})^{-s}\,\Gamma(-2-n-s)\,\Gamma(s)}{\Gamma(-2-n)} \tag{4.12}$$

$$= \frac{4}{\Gamma(4+n)\,\Gamma(-2-n)} \int_0^1 \frac{(1-t)^{3+n}\,t^{-3-n}}{(-2+t)^2}\,dt \;\; \text{for } -4 < Re(n) < -2 \tag{4.13}$$

$$= \frac{1}{\Gamma(4+n)\,\Gamma(-2-n)} \int_0^{\infty} \frac{t^{3+n}}{(\frac{1}{2}+t)^2}\,dt \;\; \text{for } -4 < Re(n) < -2 \tag{4.14}$$

$$= \frac{1}{\Gamma(-2-n)} \int_0^{\infty} e^{-t}\,t^{-3-n}{}_1F_1\left(2;2;\frac{t}{2}\right)dt \;\; \text{for } 2 + Re(n) < 0 \tag{4.15}$$

If $a = c = n$, then

$${}_2F_1\left[\begin{matrix} n, & -2n\,; & \frac{1}{2} \\ n & ; & \end{matrix}\right] = \left(\frac{\Gamma(1-n)\,\Gamma(n)\,P_{-n}^{(-1+n,-2n)}(0)}{\Gamma(0)}\right). \tag{4.16}$$

$$= \sum_{k=0}^{\infty} \frac{{}_2F_1(k+n,k-2n;k+n;z_0)(0)_k(\frac{1}{2}-z_0)^k}{k!} \quad \text{for(not}(z_0 \in R \text{ and } 1 \le z_0 \le \infty)) \tag{4.17}$$

$$= \frac{1}{\Gamma(-2n)} \int_0^{\infty} e^{-t}\,t^{-1-2n}{}_1F_1\left(n;n;\frac{t}{2}\right)dt \quad \text{for } Re(n) < 0 \tag{4.18}$$

5. Derivation of the formulae

Applying computational Technique the formulae can be established.

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