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πgη-CLOSED SETS IN BITOPOLOGICAL SPACES

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ABSTRACT
In this paper, a new class of sets namely πgη-closed, πgη-closure of a set and πgη-neighbourhood in bitopological spaces are introduced and some of their basic properties are discussed. The relationships among closed, α-closed, s-closed, η-closed, gη-closed and other generalized closed sets are investigated. Several examples are provided to illustrate the behavior of these new class of sets.
KEYWORDS : *πgη-closed and πgη-open sets; πgη-closure, πgη-neighbourhood.*
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1. INTRODUCTION

A triplet $(X, \Im_1, \Im_2)$, where X is a non-empty set and $\Im_1$ and $\Im_2$ are topologies on X is called a bitopological space. In 1963, Kelly [**6**] initiated the study of such spaces. In 1981, Bose [**1**] introduced the concepts of semi open sets in bitopological spaces. Bose and Sinha [**2**] introduced the notion of regular open sets in bitopological spaces. In 1986, Fukutake [**4**] introduced the concept of g-closed sets in bitopological spaces. In 2004, Sheik John and Sundaram [**11**] introduced pairwise π-open sets. In 2005, El-Tantawi and Abu-Donia [**3**] introduced the notion of αg-closed sets. In 2008, Khedr [**7**] introduced sg-closed sets. In 2009, Navalagi, [**8**] introduced the notion of $g^\#$-closed sets. In 2012, Veronica et. al [**10**] introduced the concept of g**-closed sets in bitopological spaces. Neelamegarajan and Jamal [**9**] introduced generalization of α-open sets in bitopological spaces. In 2014, Imran [**5**] introduced the notion of gα*-closed sets. In 2019, Subbulakshmi et. al [**13, 14**] introduced and investigated η-open and gη-closed sets. In 2020, Sivanthi [**12**] introduced the concept of πg-closed sets in bitopological spaces. Sumathi et al. [**15**] introduced the notion of gη-closed sets in bitopological spaces. In this paper, a new class of sets called πgη-closed and πgη-open sets, πgη-closure of a set and πgη-neighbourhoods in bitopological spaces and some of their basic properties are studied.

2. PRELIMINARIES

Definition 2.1. A subset A of a bitopological space $(X, \Im_1, \Im_2)$ is called
(i) $\boldsymbol{\Im_1\Im_2}$ **regular open set** [2] if $A = \Im_2int(\Im_1cl(A))$, $\Im_1\Im_2$regular closed set if $A = \Im_2cl(\Im_1int(A))$.
(ii) $\boldsymbol{\Im_1\Im_2}$ **semi-open set** [1] if $A \subset \Im_2cl(\Im_1int(A))$, $\Im_1\Im_2$semi-closed set if $\Im_2int(\Im_1cl(A)) \subset A$.
(iii) $\boldsymbol{\Im_1\Im_2}$ **α-open set** [9] if $A \subset \Im_1int(\Im_2cl(\Im_1int(A)))$, $\Im_1\Im_2$α-closed set if $\Im_1cl(\Im_2int(\Im_1cl(A))) \subset A$.
(iv) $\boldsymbol{\Im_1\Im_2}$**η-open set** [13] if $A \subset \Im_1int(\Im_2cl(\Im_1int(A))) \cup \Im_2cl(\Im_1int(A))$, $\Im_1\Im_2$η-closed set if $\Im_1cl(\Im_2int(\Im_1cl(A))) \cap \Im_2int(\Im_1cl(A)) \subset A$.

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(v) The finite union of $\Im_1\Im_2$-regular open sets is **$\Im_1\Im_2\pi$-open set** [**12**]. The complement of a $\Im_1\Im_2\pi$-open set is said to be $\Im_1\Im_2$-π-closed.

Definition 2.2. A subset A of a bitopological space (X, $\Im_1$, $\Im_2$) is called
(i) **$\Im_1\Im_2$ g-closed set** [**4**] if $\Im_2$cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$-open in X.
(ii) **$\Im_1\Im_2$ g*-closed set** [**11**] if $\Im_2$-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$-g-open in X.
(iii) **$\Im_1\Im_2$ g**-closed set** [**10**] if $\Im_2$-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$-g*-open in X.
(iv) **$\Im_1\Im_2$ gα-closed set** [**9**] if $\Im_2\alpha$-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$ α-open in X.
(v) **$\Im_1\Im_2$ αg-closed set** [**3**] if $\Im_2$cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$-open in X.
(vi) **$\Im_1\Im_2$ gα*-closed set** [**5**] if $\Im_2$cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1\alpha$-open in X.
(vii) **$\Im_1\Im_2$ -g$^{\#}$-closed set** [**8**] if $\Im_2$-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$ αg-open in X.
(viii) **$\Im_1\Im_2$ gs-closed set** [**4**] if $\Im_2$s-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$-open in X.
(ix) **$\Im_1\Im_2$ sg-closed set** [**7**] if $\Im_2$s-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$ semi-open in X.
(x) **$\Im_1\Im_2$ gη-closed set** [**14, 15**] if $\Im_2\eta$cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$-open in X.
(xi) **$\Im_1\Im_2$ πg-closed set** [**12**] if $\Im_2$cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1$ π-open in X.

3. $\Im_1\Im_2\pi$gη-CLOSED SETS

Definition 3.1. A subset A of a bitopological space (X, $\Im_1$, $\Im_2$) is called **$\Im_1\Im_2\pi$gη-closed** if $\Im_2\eta$-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_1\pi$-open in X.

The family of all $\Im_1\Im_2$-closed sets in a bitopological space (X, $\Im_1$, $\Im_2$) is denoted by $\Im_1\Im_2\pi$gη C(X, $\Im_1$, $\Im_2$)

Theorem 3.2. Every $\Im_2$-closed set is $\Im_1\Im_2\pi$gη-closed.
Proof. Let A be any $\Im_2$-closed set in (X, $\Im_1$, $\Im_2$) and A $\subset$ U, where U is $\Im_1\pi$-open. Since every $\Im_2$-closed set is $\Im_2\eta$-closed, so $\Im_2\eta$-cl(A) $\subset$ $\Im_2$cl(A) = A. Therefore, $\Im_2\eta$-cl(A) $\subset$ A $\subset$ U. Hence A is $\Im_1\Im_2\pi$gη-closed set.

Theorem 3.3. Every $\Im_2$-semi-closed set is $\Im_1\Im_2\pi$gη-closed.
Proof. Let A be any $\Im_2$-semi-closed set in (X, $\Im_1$, $\Im_2$) and A $\subset$ U, where U is $\Im_1\pi$-open. Since every $\Im_2$semi-closed set is $\Im_2\eta$-closed, so $\Im_2\eta$-cl(A) $\subset$ $\Im_2$scl(A) = A. Therefore $\Im_2\eta$-cl(A) $\subset$ A $\subset$ U. Hence A is $\Im_1\Im_2\pi$gη-closed set.

Theorem 3.4. Every $\Im_2\alpha$-closed set is $\Im_1\Im_2$ πgη-closed.
Proof. Let A be any $\Im_2\alpha$-closed set in (X, $\Im_1$, $\Im_2$) and A $\subset$ U, where U is $\Im_1\pi$-open. Since every $\Im_2\alpha$-closed set is $\Im_2\eta$-closed, so $\Im_2\eta$-cl(A) $\subset$ $\Im_2\alpha$-cl(A) = A. Therefore $\Im_2\eta$-cl(A) $\subset$ A $\subset$ U. Hence A is $\Im_1\Im_2\pi$gη-closed set.

Theorem 3.5. Every $\Im_2$regular-closed set is $\Im_1\Im_2\pi$gη-closed.
Proof. Let A be any $\Im_2$regular-closed set in (X, $\Im_1$, $\Im_2$) and A $\subset$ U, where U is $\Im_2\pi$-open. Since every $\Im_2$-regular closed set is $\Im_2$-closed. So, by **Theorem 3.2**, A is $\Im_1\Im_2\pi$gη-closed set.

Theorem 3.6. Every $\Im_2\eta$-closed set is $\Im_1\Im_2$gη-closed.
Proof. Let A be any $\Im_2\eta$-closed set in (X, $\Im_1$, $\Im_2$) and A $\subset$ U, where U is $\Im_1\pi$-open. Since A is $\Im_2\eta$- closed. Therefore $\Im_2\eta$-cl(A) = A $\subset$ U. Hence A is $\Im_1\Im_2\pi$gη-closed set.

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Theorem 3.7. Every $\Im_1\Im_2$gη-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2$gη-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$η-cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open and since every $\Im_1$π-open set is $\Im_1$-open. Given that A is $\Im_1\Im_2$gη-closed set such that $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.

Theorem 3.8. Every $\Im_1\Im_2$g-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2$g-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open and since every $\Im_1$π-open set is $\Im_1$-open. Since every $\Im_2$-closed set is $\Im_2$-η-closed, so $\Im_2$η-cl(A) ⊂ $\Im_2$cl(A) ⊂ U. Therefore $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.
Theorem 3.9. Every $\Im_1\Im_2$πg-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2$πg-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open. Since every $\Im_2$-closed set is $\Im_2$-η-closed, so $\Im_2$η-cl(A) ⊂ $\Im_2$cl(A) ⊂ U. Therefore $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.

Theorem 3.10. Every $\Im_1\Im_2$g*-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2$g*-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open and since every $\Im_1$π-open set is $\Im_1$g-open. Given that A is $\Im_1\Im_2$g*-closed set such that $\Im_2$cl(A) ⊂ U. But we have $\Im_2$η-cl(A) ⊂ $\Im_2$cl(A) ⊂ U. Therefore $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.

Theorem 3.11. Every $\Im_1\Im_2$g**-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2$g**-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open and since every $\Im_1$π-open set is $\Im_1$g*-open. Given that A is $\Im_1\Im_2$g**-closed set such that $\Im_2$cl(A) ⊂ U. But we have $\Im_2$η-cl(A) ⊂ $\Im_2$cl(A) ⊂ U. Therefore $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.

Theorem 3.12. Every $\Im_1\Im_2g^{\#}$-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2g^{\#}$-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open and since every $\Im_1$π-open set is $\Im_1$αg-open. Given that A is $\Im_1\Im_2g^{\#}$-closed set such that $\Im_2$cl(A) ⊂ U. But we have $\Im_2$η-cl(A) ⊂ $\Im_2$cl(A) ⊂ U. Therefore $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.

Theorem 3.13. Every $\Im_1\Im_2$gs-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2$gs-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$s-cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open and since every $\Im_1$π-open set is $\Im_1$-open. Given that A is $\Im_1\Im_2$gs-closed set such that $\Im_2$s-cl(A) ⊂ U. But we have $\Im_2$η-cl(A) ⊂ $\Im_2$s-cl(A) ⊂ U. Therefore $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.

Theorem 3.14. Every $\Im_1\Im_2$sg-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2$sg-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$s-cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open and since every $\Im_1$π-open set is $\Im_1$s-open. Given that A is $\Im_1\Im_2$sg-closed set such that $\Im_2$s-cl(A) ⊂ U. But we have $\Im_2$η-cl(A) ⊂ $\Im_2$s-cl(A) ⊂ U. Therefore $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.

Theorem 3.15. Every $\Im_1\Im_2$αg-closed set is $\Im_1\Im_2$πgη-closed.
Proof. Let A be any $\Im_1\Im_2$αg-closed set in (X, $\Im_1$, $\Im_2$) then $\Im_2$α-cl(A) ⊂ U whenever A ⊂ U, where U is $\Im_1$π-open and since every $\Im_1$π-open set is $\Im_1$-open. Given that A is $\Im_1\Im_2$αg-closed set such that $\Im_2$α-cl(A) ⊂ U. But we have $\Im_2$η-cl(A) ⊂ $\Im_2$α-cl(A) ⊂ U. Therefore $\Im_2$η-cl(A) ⊂ U. Hence A is $\Im_1\Im_2$πgη-closed set.

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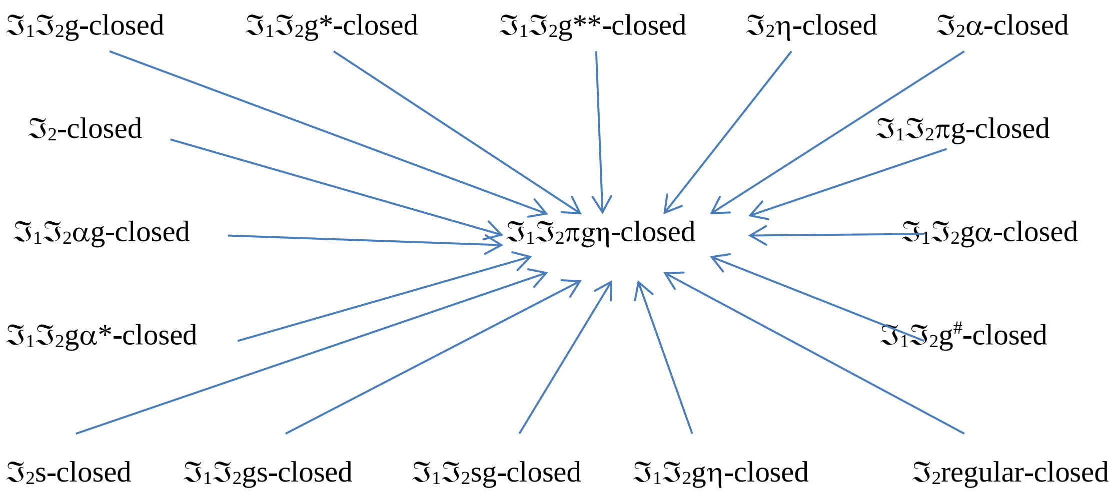
Theorem 3.16. Every $\Im_1\Im_2 g\alpha$-closed set is $\Im_1\Im_2\pi g\eta$-closed.
Proof. Let A be any $\Im_1\Im_2 g\alpha$-closed set in $(X, \Im_1, \Im_2)$ then $\Im_2\alpha$-cl(A) $\subset$ U whenever A $\subset$ U, where U is $\Im_1\pi$-open and since every $\Im_1\pi$-open set is $\Im_1\alpha$-open. Given that A is $\Im_1\Im_2 g\alpha$-closed set such that $\Im_2\alpha$-cl(A) $\subset$ U. But we have $\Im_2\eta$-cl(A) $\subset$ $\Im_2\alpha$-cl(A) $\subset$ U. Therefore $\Im_2\eta$-cl(A) $\subset$ U. Hence A is $\Im_1\Im_2\pi g\eta$-closed set.

Theorem 3.17. Every $\Im_1\Im_2 g\alpha^*$-closed set is $\Im_1\Im_2\pi g\eta$-closed.
Proof. Let A be any $\Im_1\Im_2 g\alpha^*$-closed set in $(X, \Im_1, \Im_2)$ then $\Im_2$cl(A) $\subset$ U whenever A $\subset$ U, where U is $\Im_1\pi$-open and since every $\Im_1\pi$-open set is $\Im_1\alpha$-open. Given that A is $\Im_1\Im_2 g\alpha^*$-closed set such that $\Im_2$cl(A) $\subset$ U. But we have $\Im_2\eta$-cl(A) $\subset$ $\Im_2$cl(A) $\subset$ U. Therefore $\Im_2\eta$-cl(A) $\subset$ U. Hence A is $\Im_1\Im_2\pi g\eta$-closed set.

Remark 3.18. We have the following implications for the properties of subsets:



Where none of the implications is reversible as can be seen from the following examples:

Example 3.19. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b, d}, {a, b, d}} and $\Im_2$ = {X, ϕ, {b}, {c, d}, {b, c, d}}. Then the set A = {b} is $\Im_1\Im_2\pi g\eta$-closed but not $\Im_2$closed.

Example 3.20. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b, d}, {a, b, d}} and $\Im_2$ = {X, ϕ, {b}, {c, d}, {b, c, d}}. Then the set A = {c} is $\Im_1\Im_2\pi g\eta$-closed but not $\Im_2$semi-closed.

Example 3.21. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b, d}, {a, b, d}} and $\Im_2$ = {X, ϕ, {b}, {c, d}, {b, c, d}}. Then the set A = {b} is $\Im_1\Im_2$ $\pi g\eta$-closed but not $\Im_2\alpha$-closed.

Example 3.22. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b, d}, {a, b, d}} and $\Im_2$ = {X, ϕ, {b}, {c, d}, {b, c, d}}. Then the set A = {c, d} is $\Im_1\Im$ $\pi g\eta$-closed but not $\Im_2$regular closed.

Example 3.23. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b, d}, {a, b, d}} and $\Im_2$ = {X, ϕ, {b}, {c, d}, {b, c, d}}. Then the set A = {b, c} is $\Im_1\Im_2\pi g\eta$-closed but not $\Im_2\eta$-closed.

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Example 3.24. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {X, ϕ, {a}, {b, d}, {a, b, d}}. Then the set A = {d} is $\Im_1\Im_2$πgη-closed but not $\Im_1\Im_2$g-closed.

Example 3.25. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {X, ϕ, {a}, {b, d}, {a, b, d}}. Then the set A = {a} is $\Im_1\Im_2$πgη-closed but not $\Im_1\Im_2$g*-closed.

Example 3.26. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {X, ϕ, {a}, {b, d}, {a, b, d}}. Then the set A = {a, d} is $\Im_1\Im_2$πgη-closed but not $\Im_1\Im_2$sg-closed.

Example 3.27. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {X, ϕ, {a}, {b, d}, {a, b, d}}. Then the set A = {a} is $\Im_1\Im_2$πgη-closed but not $\Im_1\Im_2$αg-closed.

Example 3.28. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {X, ϕ, {a}, {b, d}, {a, b, d}}. Then the set A = {a} is $\Im_1\Im_2$πgη-closed but not $\Im_1\Im_2$gα-closed.

Example 3.29. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {d}, {a, b, d}} and $\Im_2$ = {X, ϕ, {d}, {b, c}, {b, c, d}}. The set A = {d} is $\Im_1\Im_2$πgη-closed but not $\Im_2$-closed.

Remark 3.30. Let A and B be two $\Im_1\Im_2$πgη-closed sets, then their union and intersection need not be $\Im_1\Im_2$πgη-closed as shown from the following examples.

Example 3.31. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {X, ϕ, {b}, {c, d}, {b, c, d}}. Here the sets A = {b} and B = {c} are $\Im_1\Im_2$πgη-closed sets. But their union A ∪ B = {b, c} is not $\Im_1\Im_2$πgη-closed.

Example 3.32. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {c}, {a, b}, {a, b, c}} and $\Im_2$ = {X, ϕ, {a}, {b, d}, {a, b, d}}. Here the sets A = {a, c, d} and B = {b, c, d} are $\Im_1\Im_2$πgη-closed. But their intersection A ∩ B = {c, d} is not $\Im_1\Im_2$πgη-closed.

Theorem 3.33. Let A be a subset of a bitopological space (X, $\Im_1$, $\Im_2$). If A is $\Im_1\Im_2$πgη-closed, $\Im_2$ηcl(A) – A does not contain any non-empty $\Im_1$π-closed set.
Proof. Suppose that A is $\Im_1\Im_2$πgη-closed. Let M be a non-empty $\Im_1$π-closed set in X such that M ⊂ $\Im_2$ηcl(A) – A. Then A ⊂ X – M. Since A is $\Im_1\Im_2$πgη-closed set and X – M is $\Im_1$π-open, $\Im_2$ηcl(A) ⊂ X – M. That is, M ⊂ X – $\Im_2$ηcl(A). So M ⊂ (X – $\Im_2$ηcl(A)) ∩ ($\Im_2$ηcl(A) – A).Therefore M = ϕ.

Corollary 3.34. Let A be $\Im_1\Im_2$πgη-closed. Then A is $\Im_2$η-closed if and only if $\Im_2$cl(A) – A is $\Im_1$π-closed.
Proof. Suppose that A is $\Im_1\Im_2$πgη-closed and $\Im_2$η-closed. Since A is $\Im_2$η-closed, we have $\Im_2$η-cl(A) = A. Therefore, $\Im_2$η-cl(A) – A = ϕ which is $\Im_1$π-closed.

Conversely, suppose that A is $\Im_1\Im_2$πgη-closed and $\Im_2$ cl(A) – A is $\Im_1$π-closed. Since A is $\Im_1\Im_2$πgη-closed, we have $\Im_2$η-cl(A) – A contains no nonempty $\Im_1$π-closed set by **Theorem 3.33**. Since $\Im_2$η-cl(A) – A is itself $\Im_1$π-closed, we have $\Im_2$η-cl(A) – A = ϕ. Therefore, $\Im_2$η-cl(A) = A implies that A is $\Im_2$η-closed.

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Theorem 3.35. Let A and B be any two subsets of a bitopological space (X, $\Im_1$, $\Im_2$), such that A ⊂ B ⊂ $\Im_2$ηcl(A). If A is $\Im_1\Im_2$πgη-closed, then *B* is also $\Im_1\Im_2$πgη-closed.
Proof. Let B ⊂ P and *P* is $\Im_1$π-open in X. Since A ⊂ B, we have A ⊂ P. Since A is $\Im_1\Im_2$πgη-closed, we have $\Im_2$ηcl(A) ⊂ P. As B ⊂ $\Im_2$ηcl(A), $\Im_2$ηcl(B) ⊂ $\Im_2$ηcl(A). Hence $\Im_2$ηcl(B) ⊂ P. Therefore *B* is $\Im_1\Im_2$πgη-closed.

4. $\Im_1\Im_2$ πgη-OPEN SETS
Definition 4.1. A subset A of (X, $\Im_1$, $\Im_2$) is said to be **$\Im_1\Im_2$πgη-open** in X if its complement X – A is $\Im_1\Im_2$πgηclosed in (X, $\Im_1$, $\Im_2$).

Theorem 4.2. A subset A of a bitopological space (X, $\Im_1$, $\Im_2$) is $\Im_1\Im_2$πgη-open if and only if P ⊂ $\Im_2$η-int(A) whenever P ⊂ A and P is $\Im_1$π-closed in X.
Proof. Let A is $\Im_1\Im_2$πgη-open. Let P ⊂ A and P is $\Im_1$π-closed in X. Then $A^c \subset P^c$ and P^c is $\Im_1$π-open in X. Since A is $\Im_1\Im_2$πgη-open, we have A^c is $\Im_1\Im_2$πgη-closed. Hence $\Im_2$η-cl(A^c) ⊂ P^c. Since $\Im_2$η-cl(A) = ($\Im_2$η-int(A))^c. Consequently, ($\Im_2$η-int(A))^c ⊂ P^c. Therefore P ⊂ $\Im_2$η-int(A).

Conversely, suppose that P ⊂ $\Im_2$η-int(A) whenever P ⊂ A and P is $\Im_1$π-closed in X. Let A^c ⊂ Q and Q is $\Im_1$π-open in X. Then Q^c ⊂ A and Q^c is $\Im_1$π-closed in X. By hypothesis, Q^c ⊂ $\Im_2$η-int(A). That is, ($\Im_2$η-int(A))^c ⊂ Q. Therefore, $\Im_2$η-cl(A^c) ⊂ Q. Consequently A^c is $\Im_1\Im_2$πgη-closed. Hence A is $\Im_1\Im_2$πgη-open.

Theorem 4.3. Let A and B be subsets of a bitopological space (X, $\Im_1$, $\Im_2$) such that $\Im_2$η-int(A) ⊂ B ⊂ A. If A is $\Im_1\Im_2$πgη-open, then B is also $\Im_1\Im_2$πgη-open.
Proof. Suppose that A and B are subsets of a bitopological space(X, $\Im_1$, $\Im_2$) such that $\Im_2$η-int(A) ⊂ B ⊂ A, let A be $\Im_1\Im_2$πgη-open. Then $A^c \subset B^c$ ⊂ $\Im_2$η-cl(A^c). Since A^c is $\Im_1\Im_2$πgη-closed. By **Theorem 3.35**, B^c is $\Im_1\Im_2$πgη-closed in X. Therefore B is $\Im_1\Im_2$πgη-open.

5. $\Im_1\Im_2$πgη -CLOSURE
Definition 5.1. For a subset A of a bitopological space (X, $\Im_1$, $\Im_2$), the intersection of all $\Im_1\Im_2$πgη-closed sets containing A is called the **$\Im_1\Im_2$πgη-closure** of A and is denoted by $\Im_1\Im_2$πgη-cl(A). That is,

$$\Im_1\Im_2\pi g\eta\text{-cl}(A) = \cap\{M : A \subset M, M \text{ is } \Im_1\Im_2\pi g\eta\text{-closed in } X\}.$$

Remark 5.2. If A and B are any two subsets of a bitopological space (X, $\Im_1$, $\Im_2$), then
(i) $\Im_1\Im_2$πgη-cl(X) = X.
(ii) $\Im_1\Im_2$πgη-cl(ϕ) = ϕ.

Example 5.3. Let X = {a, b, c, d} with $\Im_1$ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {X, ϕ, {a}, {b, d}, {a, b, d}}. The $\Im_1\Im_2$πgη-closed sets are {X, ϕ, {a}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d},{a, b, c},{a, b, d}, {a, c, d}, {b, c, d}. Let A = {a, c}. Then $\Im_1\Im_2$πgη-cl(A) = {a, c}, $\Im_1\Im_2$πgη-cl(X) = X, $\Im_1\Im_2$πgη-cl(ϕ) = ϕ.

Remark 5.4. If A and B are any two subsets of a bitopological space (X, $\Im_1$, $\Im_2$), then
(i) A ⊂ B ⇒ $\Im_1\Im_2$πgη-cl(A) ⊂ $\Im_1\Im_2$πgη-cl(B).
(ii) $\Im_1\Im_2$πgη-cl($\Im_1\Im_2$πgη-cl(A)) = $\Im_1\Im_2$πgη-cl(A).

(iii) $\Im_1\Im_2\pi g\eta$-cl$(A \cup B) \supset \Im_1\Im_2\pi g\eta$-cl$(A) \cup \Im_1\Im_2\pi g\eta$-cl$(B)$
(iv) $\Im_1\Im_2\pi g\eta$-cl$(A \cap B) \subset \Im_1\Im_2\pi g\eta$-cl$(A) \cap \Im_1\Im_2\pi g\eta$-cl$(B)$.

Theorem 5.5. A is a nonempty subset of a bitopological space $(X, \Im_1, \Im_2)$. $x \in \Im_1\Im_2\pi g\eta$-cl$(A)$ if and only if $A \cap V \neq \phi\ \forall\ \Im_1\Im_2\pi g\eta$-open set V containing x.
Proof. A is a nonempty subset of a bitopological space $(X, \Im_1, \Im_2)$ and $x \in \Im_1\Im_2\pi g\eta$-cl$(A)$. Suppose there exists a $\Im_1\Im_2\pi g\eta$-open set V containing x such that $A \cap V = \phi$. Then $A \subset X - V$ and $X - V$ is a $\Im_1\Im_2\pi g\eta$-closed set and so $\Im_1\Im_2\pi g\eta cl(A) \subset X - V$. Therefore $x \notin V$ which is a contradiction. Hence $A \cap V \neq \phi\ \forall$ $\Im_1\Im_2\pi g\eta$-open set V containing x.

Conversely, A is a nonempty subset of a bitopological space $(X, \Im_1, \Im_2)$ and $x \in X$ is such that $A \cap V \neq \phi$ $\forall\ \Im_1\Im_2\pi g\eta$-open set V containing x. $x \notin \Im_1\Im_2\pi g\eta$-cl$(A)$.
$\Rightarrow$ There exists a $\Im_1\Im_2\pi g\eta$-closed set F such that $A \subset F$ and $x \notin F$.
$\Rightarrow$ There exists a $\Im_1\Im_2\pi g\eta$-open set $X - F$ containing x and $A \cap (X - F) = \phi$ which is a contradiction. Therefore $x \in \Im_1\Im_2\pi g\eta$-cl$(A)$.

6. $\Im_1\Im_2\pi g\eta$ -NEIGHBOURHOODS

Definition 6.1. Let X be a bitopological space and let $x \in X$. A subset N of X is said to be a **$\Im_1\Im_2\pi g\eta$-neighbourhood** of x if and only if there is a $\Im_1\Im_2\pi g\eta$-open set G such that $x \in G \subset N$.

Definition 6.2. A subset N of a bitopological space X, is called a **$\Im_1\Im_2\pi g\eta$-neighbourhood** of $A \subset X$ if and only if there exists a $\Im_1\Im_2\pi g\eta$-open set G such that $A \subset G \subset N$.

Theorem 6.3. Every neighbourhood N of $x \in X$ is a $\Im_1\Im_2\pi g\eta$-neighbourhood of $(X, \Im_1, \Im_2)$.
Proof. Let N be a neighbourhood of a point $x \in X$. To prove that N is a $\Im_1\Im_2\pi g\eta$-neighbourhood of x. By **Definition 6.2**, there exist an open set G such that $x \in G \subset N$. As every open set is $\Im_1\Im_2\pi g\eta$-open, so G is $\Im_1\Im_2\pi g\eta$-open. Therefore, we have $x \in G \subset N$. Hence N is $\Im_1\Im_2\pi g\eta$-neighbourhood of X.

Remark 6.4. In general a $\Im_1\Im_2\pi g\eta$-neighbourhood N of $x \in X$ need not to be a neighbourhood of x in X, as in the following example.

Example 6.5. Let $X = \{a, b, c, d\}$ with $\Im_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\phi = \{X, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$. The set $\{a, c\}$ is $\Im_1\Im_2\pi g\eta$-neighbourhood of the point c, since the $\Im_1\Im_2\pi g\eta$-open set $\{a, c\}$ is such that $c \in \{a, c\} \subset \{a, c\}$. However the set $\{a, c\}$ is not a neighbourhood of the point c, since no open set G exists such that $c \in G \subset \{a, c\}$.

7. CONCLUSION

In this paper, a new class of sets namely $\pi g\eta$-closed sets, $\pi g\eta$-closure of a set, $\pi g\eta$-open sets, $\pi g\eta$-neighbourhoods in bitopological spaces are studied and some of their basic properties are discussed. The relationships among closed, α-closed, s-closed, η-closed, $g\eta$-closed and other generalized closed sets are investigated. Several examples are provided to illustrate the behavior of these new class of sets. The $\Im_1\Im_2\pi g\eta$-closed set can be used to derive a new decomposition of closed map, open map, continuity, homeomorphism, and new separation axioms. This idea can be extended to topological ordered spaces, bitopological ordered spaces and fuzzy topological spaces.

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