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HIGHER-ORDER ISOPARAMETRIC RECTANGULAR ELEMENT BASED FINITE ELEMENT METHOD FOR EIGENVALUE PROBLEMS IN ELECTROMAGNETICS

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ABSTRACT

In this paper, I have presented a different approach to the finite element method for solving the eigenvalue problem in electromagnetics using the isoparametric rectangular element for the arbitrarily shaped domain. The concept behind this approach is to map the whole geometrical domain into the minimum quadrilaterals (curve edges, if any). Then, every four vertices of a quadrilateral are transformed into a master rectangular element in a natural coordinate system using the Lagrangian interpolation basis function and Jacobin matrix. This is not a new concept, but an observed fact is that we can simply achieve a high degree of accuracy using very less isoparametric elements of a high order. After this, the ***method of residual*** *and* ***Galerkin method*** *is used to solve the weak form of the Helmholtz equation on a circular waveguide using a single second-order quadrilateral element and ridged circular waveguide.*

KEYWORDS: *finite element method, fem, isoparametric formulation, Jacobian transformation, minimum isoparametric elements, quadrilateral element, Galerkin method, eigenvalue problem, electromagnetics.*

1. INTRODUCTION

The finite element method (FEM) has been widely used to solve the partial differential equation in the complex domain. But the major problem in working with FEM is dealing with the generation of mesh and connectivity of elements. In this paper, I have presented a very simple approach to partitioning the given domain into minimum numbers of quadrilateral elements. Note that each quadrilateral should have nodes as per Lagrangian quadrilateral or rectangular element that is it should have nodes 4, 9, 16… for an order 1, 2, 3… respectively. In this case, we can not use the serendipity element because we are using very fewer elements and all the nodes are lies on the edges or in this case, on the boundary of the domain so that it will not cover the interior domain. One can use another method to increase a large number of elements but it required meshing software. The objective of this paper is to use fewer elements to create complex geometry just by providing coordinates of the nodes of that quadrilateral element. For the generation of node coordinates, I have used Blender opensource package with python script one can use simple Inkscape also. The Lagrange basis functions are used for the transformation of quadrilateral into a standard square $[-1,1] \times [-1,1]$ domain and it is also used for the nodal displacement in the element [2], At least I can say that there is no need for complex meshing or any mesh generation software in this approach just coordinates of domain points are needed. The concept is given in section 2. In section 5.1 Helmholtz equation is solved on a standard circular waveguide to test this approach and in section 5.2 same problem is solved on the arbitrary circular waveguide.

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2. Mapping of the Circular Waveguide using a Single Second-Order Quadrilateral Element (9 nodes).

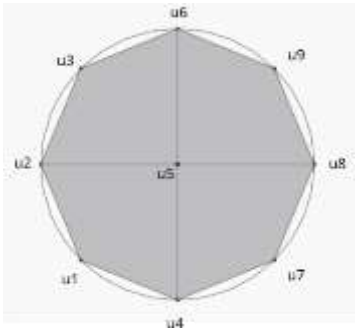

Fig. 1. Mapping of the circular domain using single Second-order quadrilateral element (9 nodes).

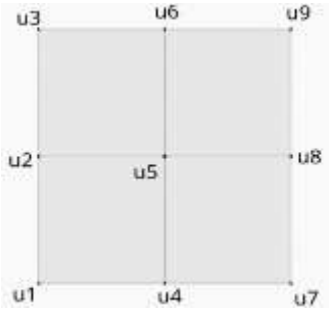

Fig. 2. Standard second-order rectangular element

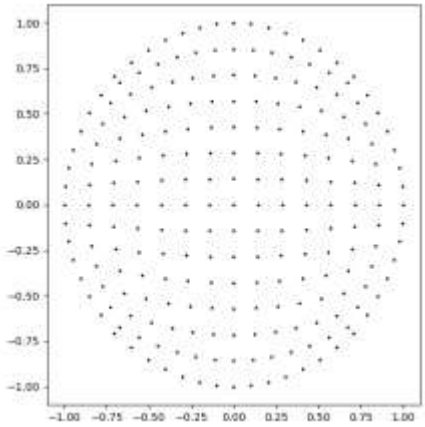

Fig. 3. The 2D plot of transformation equation of circular waveguide.

In this approach whole domain is mapped into the minimum number of elements. For a circular domain, only a quadratic element is sufficient to map exactly into a circular domain as shown in Fig.1. hence only 9 nodes will cover the whole domain into a standard quadratic rectangular element as shown in Fig. 2. the order of elements depends on the number of corners and curved edges in the geometrical domain but the number of points should be 4, 6, 9, and so on. In the above example, the second-order (p=2) shape functions are needed to transform the whole domain into a second-order master rectangular element as shown in Fig 2. note that here only a single second-order quadrilateral element is used and not 4 elements as appears in Fig.1. after isoparametric mapping using the Lagrange shape function this element exactly cover complete circle without any discretization error. The result of the circular domain is shown in Fig. 3.

To find the approximate solution of a given differential equation it is necessary to have good enough interior points in the domain. To increase the internal points we can increase the order of approximation. It is not necessary to have large points for approximating the domain. It is observed that 1 to 3 order of shape functions are sufficient to map many arbitrary-shaped domains. Once the domain is approximated, we can increase the order of the lagrangian shape function for nodal displacement to approximate the solution of the given differential equation.

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3. SHAPE FUNCTIONS

The Lagrange linear polynomial shape functions along X-axis are given by

$$l_i(x) = \prod_{j=0\ j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

And along Y-axis

$$l_i(y) = \prod_{j=0\ j\neq i}^{n} \frac{(y - y_j)}{(y_i - y_j)}$$

Taking the tensor product of the above two-liner shape function, we can obtain the 2D basis shape functions $\phi(s,t)$.

4. TRANSFORMATIONS AND JACOBIAN

Jacobin is the matrix that is useful in the transformation of one coordinate system to another; it may be Cartesian to Cartesian or Cartesian to polar etc.

4.1. 1D Case

This transformation is used in the calculation of line integral on the boundary of the domain. For line integration $f = f(x,y)$ and $x = x(s), y = y(s)$ these are the parametric equation used for the transformation of the simple horizontal line called a master element, and its endpoints distances are -1 and 1.

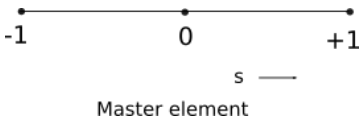

Fig. 4

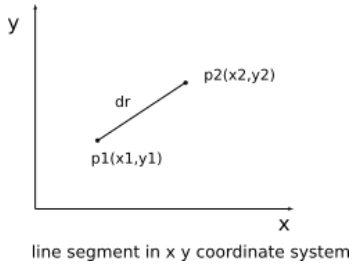

Fig. 5

To map an inclined line in the XY coordinate system to the master element, I have used the Lagrange polynomial for the transformation equation.

Let $\phi(s,t)$

$x = x_1\phi_1 + x_2\phi_2$ And $y = y_1\phi_1 + y_2\phi_2$

Here $\phi_1(s)$ and $\phi_2(s)$ are the linear shape functions or linear basis functions in the natural coordinate system, that is $\phi = \phi(s)$.

The line integration on the segment dr is written as

$$\int_a^b f(x,y)dr = \int_{-1}^{1} f\big(x(s),y(s)\big)\sqrt{(dx)^2 + (dy)^2}$$

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$$\int_a^b f(x,y)dr = \int_{-1}^{1} f\big(x(s),y(s)\big)\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2}\ ds$$

Where $\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2}$ is the Jacobin J.

For linear shape function on boundary only x or y variable present therefore

$\int_a^b f(x)dx = \int_{-1}^{1} f\big(x(s)\big)J\ ds$

Where $J = \frac{dx}{ds}$

Numerically, line integration is solved by using Gauss Quadrature integration [9]. which is

$$\int_a^b f(x)dx = \frac{b-a}{2}\sum_{i=1}^{n} w_i f\big(x(s)\big)J$$

Where, w_i are the Gauss-Quadratures weights. And note that Jacobian is evaluated at Gauss sampling points. To find the derivative of shape function ϕ, as we know that shape function ϕ is the function of s only and we have to calculate the derivative with the respective x. then

$$\frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{ds}{dx}$$

$$\therefore \frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{1}{|J|}$$

Where, $J = \frac{dx}{ds}$ is the Jacobin.

4.2. 2D Case

In this case, transformation equations are [1]

$x = \sum_{i=1}^{N} x_i\phi_i(s,t)$ …1

$y = \sum_{i=1}^{N} y_i\phi_i(s,t)$ …2

Where (x_i, y_i) the coordinates of domain points, (x,y) are the new points in the domain corresponding to natural coordinates (s,t) in the master element. Where N is the number of points in the domain.

Suppose we have a function $f(x,y)$ and in the Cartesian coordinate system and by transformation technique we got the values of x and y such that $x = x(s,t)\ and\ y = y(s,t)$. Where, s and t are horizontal and vertical axes in the natural coordinate system. Suppose we want to integrate $f(x,y)$ function, then the surface integration with transformation is given by [2]

$\int\int f(x,y)dxdy = \int\int f\big(x(s,t),y(s,t)\big)J\ dsdt$ …3

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4.2.3. Calculation of Jacobian in 2D

Let $f = f(x,y)$ and $x = x(s,t), y = y(s,t)$ According to the chain rule [2]

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Consider the above equation as a system of two linear equations with two unknown $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. In matrix form

$$\begin{pmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ \frac{dx}{dt} & \frac{dy}{dt} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial f}{\partial t} \end{pmatrix}$$

$$D = \begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ \frac{dx}{dt} & \frac{dy}{dt} \end{vmatrix}$$

$$D_x = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{dy}{ds} \\ \frac{\partial f}{\partial y} & \frac{dy}{dt} \end{vmatrix}$$

$$D_y = \begin{vmatrix} \frac{dx}{ds} & \frac{\partial f}{\partial x} \\ \frac{dx}{dt} & \frac{\partial f}{\partial y} \end{vmatrix}$$

According to the crammers rule [2]

$\frac{\partial f}{\partial x} = \frac{Dx}{D}$ and $\frac{\partial f}{\partial y} = \frac{Dy}{D}$. here $D = |J|$ is the 2D Jacobin

Once we know the Transformation and Jacobian then we can integrate and differentiate the shape functions. Using Gauss-Quadrature integration (ref. 9), equation 2 is written as

$$\int_a^b \int_c^d f(x,y)dxdy = \left(\frac{b-a}{2}\right)\left(\frac{d-c}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f\big(x(s,t), y(s,t)\big) J$$

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Note that, here jacobian is evaluated at Gauss sampling points.

Where w_i *or* w_j are Gauss-Legender weight points.

5. NUMERICAL EXAMPLES

5.1. Homogeneous circular waveguide-scalar formulation

In this section, the two-dimensional circular waveguide of unit radius is taken and the problem is solved using the Galerkin method. only a single quadratic quadrilateral element is used. Finite element stiffness matrix and mass matrix are derived in python program to calculate cutoff wavenumber and electric filed distribution are presented. The cutoff wavenumbers are shown in Table 1 and field distribution is shown in figure 4.

5.1.2. Weak formulation of Helmholtz equation.

Helmholtz equation for scalar potential ϕ and wave number k_c is

$$\nabla^2\phi + k_c^2\phi = 0 \quad \dots 1$$

Multiply the above equation by test function T and integrate over surface Ω

$$\int\int [T\nabla^2\phi + k_c^2\phi]dxdy = 0$$

$$\therefore \int\int [T\nabla.\nabla\phi + k_c^2\phi]dxdy = 0 \quad \dots 2$$

Taking integration by parts of the first part of equation 2 we get

$$\int\int T\nabla.\nabla\phi\, dxdy = -\int_\Omega (\nabla T).\nabla\phi\, dxdy + \int_{d\Omega} (\nabla\phi.\hat{n})T\, dl$$

where $\hat{n}$ is the unit normal to the circular boundary and it vanishes because, for PEC in TM mode T is zero, and for TE mode $\nabla\phi = 0$

Therefore equation 2 become

$$\int_\Omega (\nabla T).\nabla\phi\, dxdy = \int_\Omega k_c^2\phi T dxdy = 0 \quad \dots 3$$

Let trial solution, $\phi = \sum_{i=1}^{N} \phi_i N_i(x,y)$

And as per Galerkin method, $T = \sum_{i=1}^{N} N_i(x,y)$

Where $N_i(x,y)$ are Lagrange basis functions. N is the total number of nodes per element or number of basis functions used for interpolation.

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Equation 3 becomes

$$\sum_{\{i,j=1\}}^{N} \phi_i \int_{\Omega} \nabla N_i . \nabla N_j \, dxdy = k_c^2 \sum_{i}^{N} \phi_i \int_{\Omega} N_i N_j \, dxdy$$

This can be written in matrix form at the element level

$$[K_e]\{\phi_e\} = k_c^2 [M_e]\{\phi_e\}$$

$$K_e = \phi_i \sum_{i,j=1}^{N} \int_{\Omega} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) dxdy \ , \qquad N \times N \ matrix$$

$$M_e = \phi_i \sum_{i,j=1}^{N} \int_{\Omega} N_i N_j dxdy \ , \qquad N \times N \ matrix$$

After assembling, the global matrix equation is

$$[K]\{\phi\} = k_c^2 [M]\{\phi\}$$

And $\phi = N \times 1$ unknown column vector that we want to calculate.

5.1.3. Boundary condition: TE and TM Modes

For TE modes, ϕ represents the axial magnetic field H_z, and the boundary condition is the Neumann that is $\frac{d\phi}{dn} = 0$. And for TM mode $\phi = 0$ at all boundaries [4].

5.3.4. Electric filed calculation from potential ϕ

For TE mode electric field can be calculated from

$$E_x = -\frac{\partial \phi}{\partial y}$$

$$E_y = \frac{\partial \phi}{\partial x}$$

For TM mode

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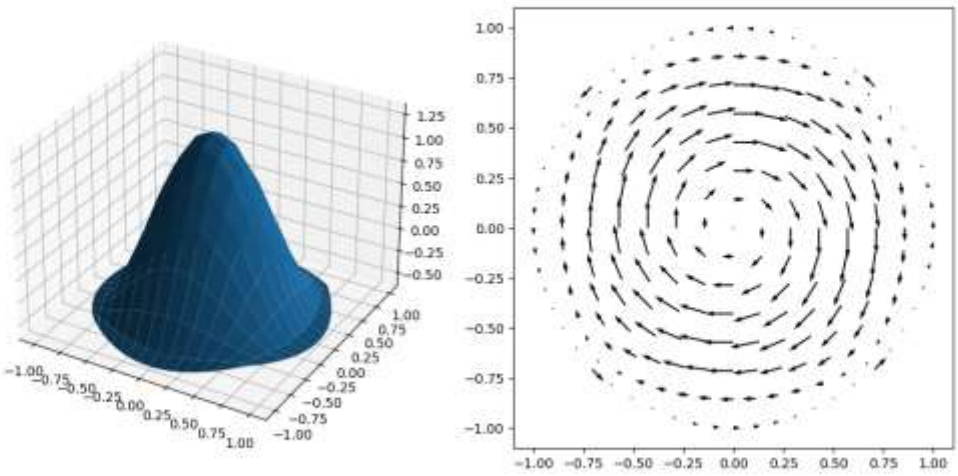
$$E_x = -Z_0 \frac{\partial \phi}{\partial x}$$

$$E_y = -Z_0 \frac{\partial \phi}{\partial y}$$

Table 1. cutoff wavenumber for circular waveguide ($k_c * radius$)

Modes	Analytical (ref 1)	FEM Solution
TE01	3.832	3.8714
TE11	1.841	1.8525
TE12	5.331	5.6209
TE21	3.054	3.0634
TE22	6.706	6.7816
TM01	2.405	2.4198
TM11	3.832	3.8579
TM12	7.016	7.08
TM21	5.135	5.1922
TM22	8.417	8.8426



(a) TE01 mode

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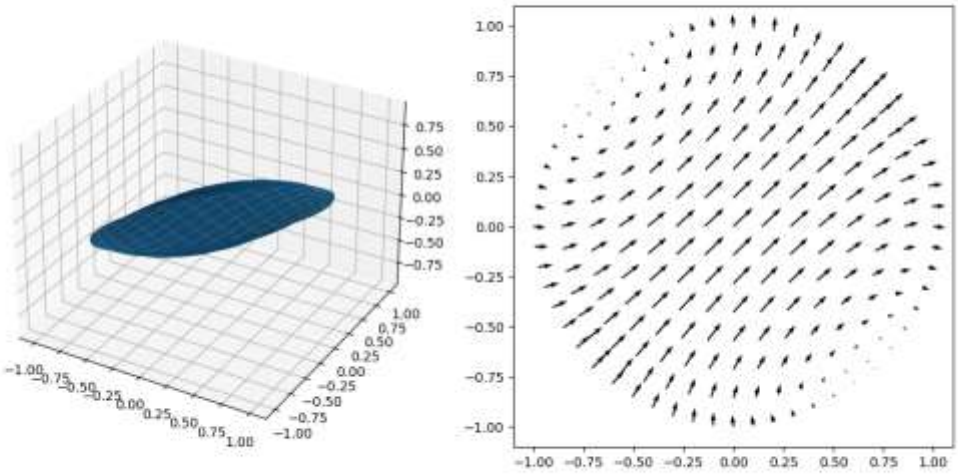
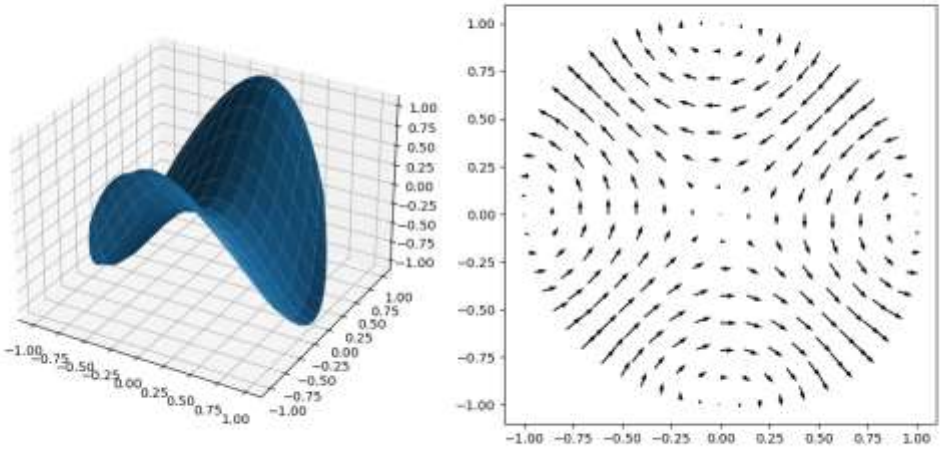


(b) TE11 mode



(c) TE11 mode

Fig 6. Potential and electric field distribution in circular waveguide for TE mode.

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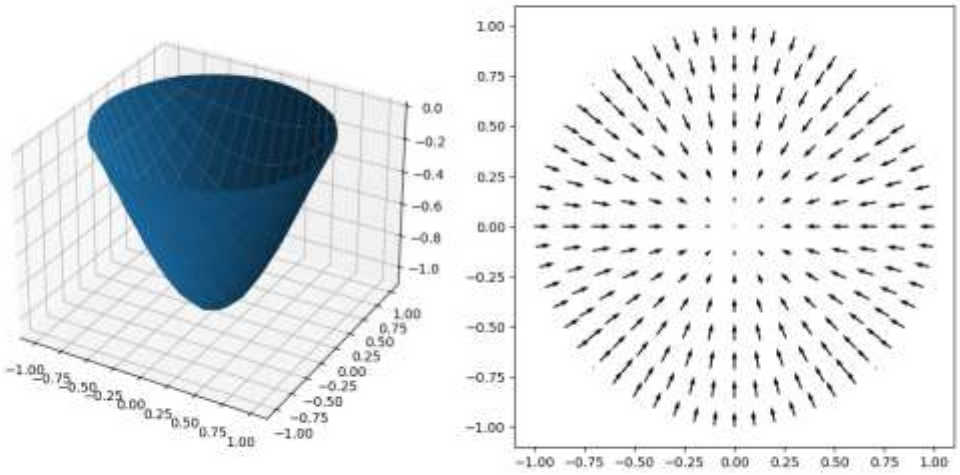
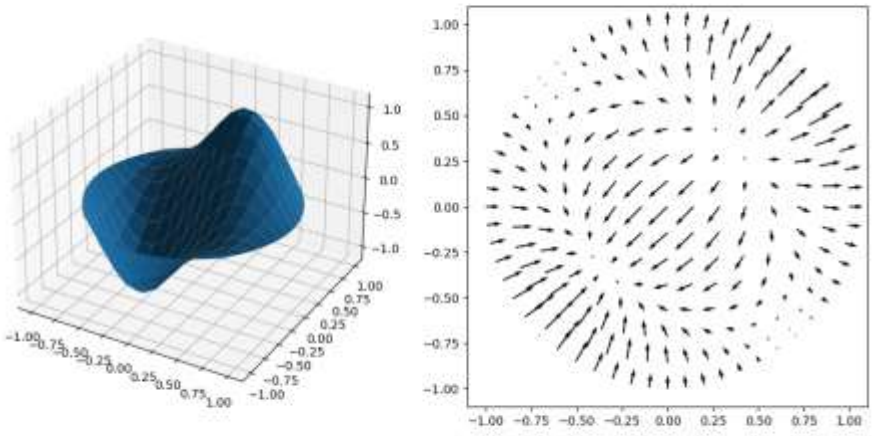


(d) TM01 mode



(e) TM11 mode

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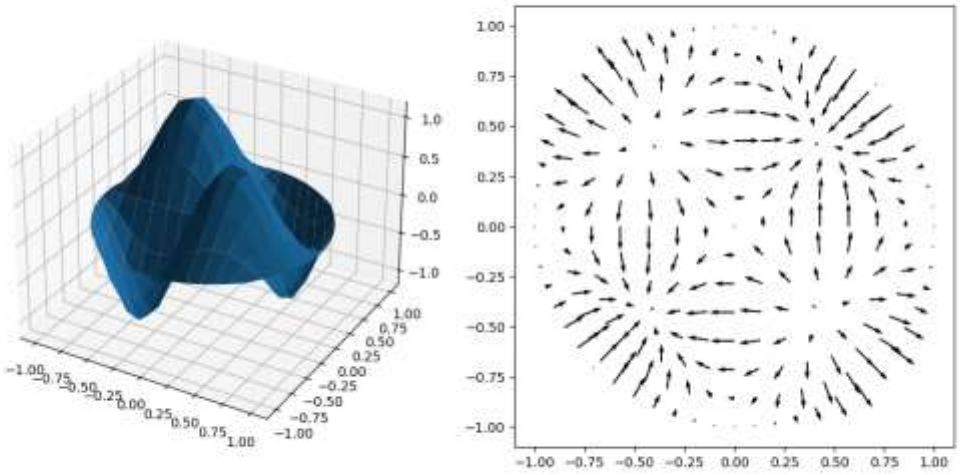

(f) TM21 mode

Fig. 7. Potential and magnetic filed distribution for TM mode

5.2. Arbitrary-shaped circular waveguide.

The same above problem was solved on an arbitrarily shaped waveguide as shown in fig.8. this domain is formed by displacing the left and right side nodes by 0.5unit inside the circle. The actual domain formed is shown in Fig.9. cutoff wavenumber for TE modes are shown in Table 2 and TE field distributions are shown in Fig.10.

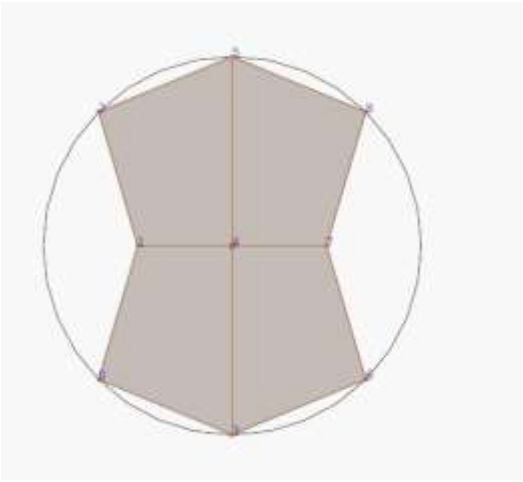

Fig. 8. Single quadrilateral element

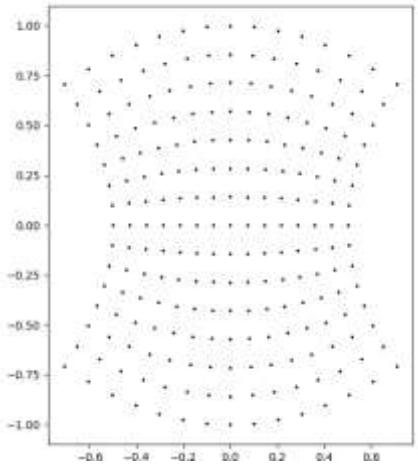

Fig. 9. Actual domain after transformation.

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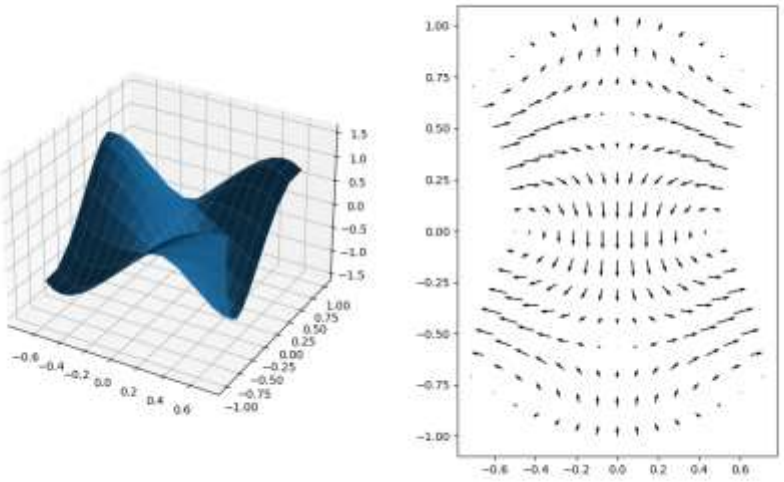
Table 2. Cutoff Wavenumber for Arbitrary Shaped Circular Waveguide ($k_c * radius$)

Modes	Standard circular waveguide	FEM Solution
TE01	3.8714	4.6708
TE11	1.8525	2.7958
TE12	5.6209	6.7713
TE21	3.0634	3.1256
TE22	6.7816	7.2025



(g) TE01 mode

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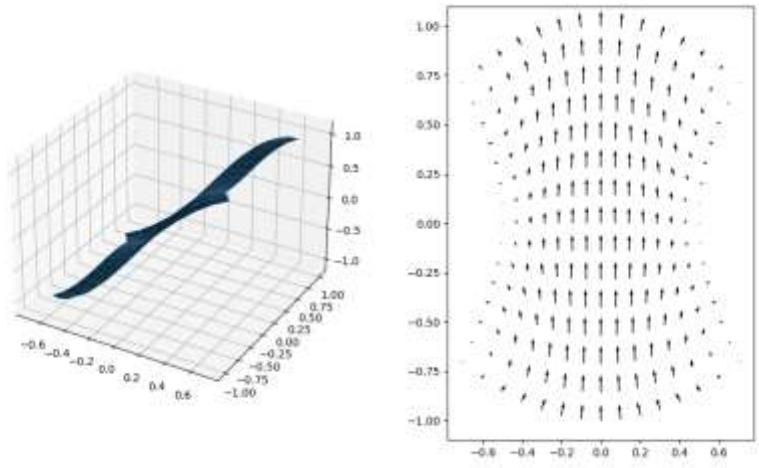
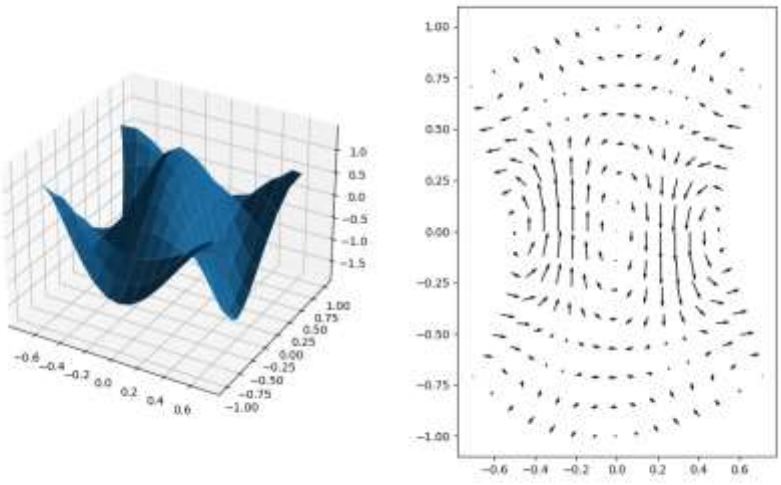


(h) TE11 mode



(i) TE12 mode

4. CONCLUSION

The technique of Iosparmetic formulation in partitioning the complex domain into minimum isoparametric elements is found to be accurate and simple. As evidence that the approach has substantial potential for applications, we note that it has been fairly easy to partition the domain into isoparametric elements just by providing the coordinates of domain points. It is observed that we can generate many complex shapes using minimum elements.

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