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ON THE DIOPHANTINE EQUATION $3^x + 7^y = z^2$

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ABSTRACT

In this paper i have shown that the Diophantine equation $3^x + 7^y = z^2$ has exactly two solutions in non-negative integers $(x, y, z) = (1, 0, 2)$ and $(2, 1, 4)$.

KEYWORDS: Exponential Diophantine equation, Catalan’s conjecture.

I. INTRODUCTION

There are a lot of studies about Diophantine equations of the type $a^x + b^y = c^z$ by a number of mathematicians in the field of number theory. In 1999, Cao [1], proved that this equation has at most one solution with $c>1$. In 2012, Peker and Cenberci [2] suggested that the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. However, in the same year, Sroysang [3] proved that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution in $(x, y, z)$ which is $(1, 0, 3)$. Same author [4, 5] also solved the Diophantine equations $8^x + 13^y = z^2$ and $8^x + 7^y = z^2$, respectively. He founded that the solution of these equations is $(1, 0, 3)$ in non-negative integers $(x, y, z)$. Robago [6] showed that the Diophantine equation $8^x + 17^y = z^2$ has four solutions in non-negative integers $(x, y, z)$. Further, several Diophantine equations of different types have been studied by different workers [7, 8, 9, 10, 11, 12, 13, 14]. In the year 2014 A. Suvarnamani [12] solved $p^x + q^y = z^2$ where $p$ is an odd prime number for which $q-p=2$ and $x, y$ are non-negative integers. Here I have made an attempt to solve the new Diophantine equation containing the prime numbers $p=3, q=7$ and $q-p=4$ hitherto uninvestigated by any researcher to the best of my knowledge, and have found that it has two solutions in non-negative integers $(x, y, z) = (1, 0, 2)$ and $(2, 1, 4)$.

II. PRELIMINARIES

The Catalan’s conjecture is an important well known conjecture and plays an important role in solving Diophantine equations. According to this conjecture, $(3, 2, 2, 3)$ is solution $(a, x, b, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a, b, x$ and $y$ are integers with $\min \{a, b, x, y\} > 1$. This conjecture was proved by Mihaiescu in 2004 [10].

2.1 Proposition:

$(3, 2, 2, 3)$ is solution $(a, x, b, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a, b, x$ and $y$ are integers with $\min \{a, b, x, y\} > 1$.

Now we will prove two lemmas by proposition 2.1.

2.1 Lemma: $(1, 2)$ a unique solution $(x, z)$ for the Diophantine equation $3^x + 1 = z^2$, where $x, z$ are non-negative integers.

Proof: let $x$ and $z$ be non-negative integers such that $3^x + 1 = z^2$.

First we consider the case $x=0$ and $z=0$.

If $x=0$ then $z^2 = 2$, which is impossible.

If $z=0$ then $3^x = -1$, which is impossible.

Now we consider the case $x, z>0$. Then $3^x + 1 = z^2$ or $3^x = z^2 - 1$.

Then $3^x = (z - 1)(z + 1)$.

Thus $(z - 1) = 3^u$, where $u$ is a non-negative integer.

Then $(z + 1) = 3^{x-u}$.
There are two possibilities for this equation

3x + 7y = z^2

Part (i): two parts i.e. part (i) and part (ii).

Case (ii) when x is odd i.e., x=2k+1 for some integer k, then

2y = z^2 - 7y

This implies that k=0 and z^2 = 2, which is not solvable.

Solving the second set of equations, 2(2, 7k) = 1 - 3^k = (3, 72k) this gives

1 - 3^k = 4.7^k + (3, 72k)

which implies that k=0 and 3^k = 2, which is not solvable.

Solving the first set of equations, we get

2(2, 7k) = 3^k + (3, 72k) - 1

i.e.3^k - 1 = (4.7^k) - (3, 72k) = 7k(4 - (3.7k))

which implies that k=0 and 3^k = 2, which is not solvable.

Solving the second set of equations, 2(2, 7k) = 1 - 3^k = (3, 72k) this gives

1 - 3^k = 4.7^k + (3, 72k)

= 7k(4 + (3.7k)) which implies that k=0 and 1 - 3^k = 7 or 3^k = -6 which is not solvable for x.

Part (ii) again we have, 3^k + 72k+1 = z^2

Or 3^k + (36 - 29)72k = z^2

i.e. 3^k + (-29)72k = z^2 - (36)(72k)

= (z - (6)(7k))(z + (6)(7k))

There are two possibilities for this equation

\[
\begin{align*}
\text{Or } z + 6.7^k &= 3^k - (29, 72k) \\
(z - 6.7^k) &= 3^k - (29, 72k)
\end{align*}
\]

III. MAIN RESULT

3.1 Theorem: The Diophantine equation

3x + 7y = z^2

has two solutions in non-negative integers x, y, z.

Proof: Let x, y and z be non-negative integers such that

(x, y, z) = (1, 0, 2) and (2, 1, 4)

No solutions when x is zero by lemma 2.2.

From lemma 2.2, x ≥ 1. Now we will divide y into two cases when x ≥ 1.

Case (i) if y is even i.e. y=2k, for some positive integer k, then

3x = z^2 - 7y = z^2 - 7^k = (z - 7^k)(z + 7^k)

This gives

z = 7^k + 1 which implies that the equation

z + 7^k = 3^k is not solvable until k=0.

So that (x, y, z) = (1, 0, 2).

Case (ii) when y is odd i.e. let y=2k+1, where k is a non-negative integer. We will divide this case into two parts i.e. part (i) and part (ii).

Part (i): 3^k + 72k+1 = z^2 or 3^k + (3 + 4)72k = z^2

i.e. 3^k + 72k = z^2 - 4.7^k(z - 2.7k) which gives

There are two possibilities for this equation

\[
\begin{align*}
\text{Or } &z - 2.7^k = 1 \\
z + 2.7^k &= 3^k + 3.72k \\
z + 2.7^k &= 1
\end{align*}
\]

Solving the first set of equations we get

2(2, 7k) = 3^k + (3, 72k) - 1

i.e.3^k - 1 = (4.7^k) - (3, 72k) = 7k(4 - (3.7k))

which implies that k=0 and 3^k = 2, which is not solvable.

Solving the second set of equations, 2(2, 7k) = 1 - 3^k = (3, 72k) this gives

1 - 3^k = 4.7^k + (3, 72k)

= 7k(4 + (3.7k)) which implies that k=0 and 1 - 3^k = 7 or 3^k = -6 which is not solvable.

We consider the case when y, z>0. Then

From lemma 2.2, we have x=0 then

This gives

No solutions when x is zero by lemma 2.2.

From lemma 2.2, x ≥ 1. Now we will divide y into two cases when x ≥ 1.

Case (i) if y is even i.e. y=2k, for some positive integer k, then

3x = z^2 - 7y = z^2 - 7^k = (z - 7^k)(z + 7^k)

This gives

z = 7^k + 1 which implies that the equation

z + 7^k = 3^k is not solvable until k=0.

So that (x, y, z) = (1, 0, 2).

Case (ii) when y is odd i.e. let y=2k+1, where k is a non-negative integer. We will divide this case into two parts i.e. part (i) and part (ii).

Part (i): 3^k + 72k+1 = z^2 or 3^k + (3 + 4)72k = z^2

i.e. 3^k + 72k = z^2 - 4.7^k(z - 2.7k) which gives

There are two possibilities for this equation

\[
\begin{align*}
\text{Or } &z - 2.7^k = 1 \\
z + 2.7^k &= 3^k + 3.72k \\
z + 2.7^k &= 1
\end{align*}
\]
Hence \((x, y, z) = (1, 0, 2)\) and \((2, 1, 4)\) are the solutions of the Diophantine equation \(3^x + 7^y = z^2\) for non-negative integers \(x, y\) and \(z\).

**3.2 Corollary:** The Diophantine equation \(3^x + 7^y = w^{2n}\) has two solutions \((x, y, w, n) = (1, 0, 2, 1)\) and \((2, 1, 4, 2)\) where \(x, y, w, n > 0\) are non-negative integers.

**Proof:** suppose that \(x, y\) and \(z\) non-negative integers such that \(3^x + 7^y = w^n\).

Let \(z = w^n\) then equation \((I)\) becomes \(3^x + 7^y = z^2\) then by theorem 3.1, we have \((x, y, z) = (1, 0, 2)\) and \((2, 1, 4)\).

When \((x, y, z) = (1, 0, 2)\), we have \(w = 2, n = 1\) \((x, y, w, n) = (1, 0, 2)\).

When \((x, y, z) = (2, 1, 4)\), we have \(w = 2, n = 2\) \((x, y, w, n) = (2, 1, 2, 2)\). This completes the proof.

**IV. OPEN PROBLEM**

Let \(p\) and \(q\) be positive prime numbers. We may ask for the set of all solutions \((x, y, z)\) for the Diophantine Equation \(p^x + q^y = z^2\) where \(x, y\) and \(z\) are non-negative integers.

**V. CONCLUSION**

In this theorem, we have solved the Diophantine Equation \(3^x + 7^y = z^2\), where \(p=3, q=7\) are prime numbers and \(q-p=4\). We have shown that the entitled equation has two solutions \((x, y, z) = (1, 0, 2)\) and \((2, 1, 4)\).

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**AUTO-BIOGRAPHY**

Chikkavarapu Gnanendra Rao,(born on 31-1-83 in Nuzvid, Andhra Pradesh) completed post-graduation in mathematics from P.B.Siddartha College of arts and science in Vijayawada in the year 2005. Presently he is working as assistant professor of mathematics in Sri Sarathi Institute of Engineering and Technology, Nuzvid. He was qualified in GATE 2013 and APSET-2012.