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ON THE DIOPHANTINE EQUATION $3^x + 7^y = z^2$

Chikkavarapu Gnanendra Rao
Assistant Professor of mathematics, Department S&H, SSIET, Nuzvid, Andhra Pradesh, India.

ABSTRACT
In this paper i have shown that the Diophantine equation $3^x + 7^y = z^2$ has exactly two solutions in non-negative integers (x, y, z) = (1, 0, 2) and (2, 1, 4).
KEYWORDS: *Exponential Diophantine equation, Catalan's conjecture.*

I. INTRODUCTION

There are a lot of studies about Diophantine equations of the type $a^x + b^y = c^z$ by a number of mathematicians in the field of number theory. In 1999, Cao [1], proved that this equation has at most one solution with c>1.In 2012, Peker and Cenberci [2] suggested that the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. However, in the same year, Sroysang [3] proved that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution in (x, y, z) which is (1, 0, 3). Same author [4, 5] also solved the Diophantine equations $8^x + 13^y = z^2$ and$8^x + 7^y = z^2$, respectively. He founded that the solution of these equations is (1, 0, 3) in non-negative integers (x, y, z).Robago [6] showed that the Diophantine equation$8^x + 17^y = z^2$ has four solutions in non-negative integers (x, y, z).Further, several Diophantine equations of different types have been studied by different workers [7, 8, 9, 10,11, 12,13,14].In the year 2014 A.Suvarnamani[12] solved $p^x + q^y = z^2$ where p is an odd prime number for which q-p=2 and x, y are non-negative integers. Here i have made an attempt to solve the new Diophantine equation containing the prime numbers p=3, q=7 and q-p=4 hitherto uninvestigated by any researcher to the best of my knowledge, and have found that it has two solutions in non-negative integers (x, y, z) = (1, 0, 2) and (2, 1, 4).

II. PRELIMINARIES

The Catalan's conjecture is an important well known conjecture and plays an important role in solving Diophantine equations. According to this conjecture, (3, 2, 2, 3) is solution (a, x, b, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with min {a, b, x, y} > 1. This conjecture was proved by mihailescu in 2004[10].

2.1 Proposition:
(3, 2, 2, 3) is solution (a, x, b, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with min {a, b, x, y} > 1.

Now we will prove two lemmas by proposition 2.1.

2.1 Lemma: (1, 2) a unique solution (x, z) for the Diophantine equation $3^x + 1 = z^2$, where x, z are non-negative integers.

Proof: let x and z be non-negative integers such that $3^x + 1 = z^2$---- (1)
First we consider the case x=0 and z=0.
If x=0 then$z^2 = 2$, which is impossible.
If z=0 then$3^x = -1$, which is impossible.
Now we consider the case x, z>0. Then $3^x + 1 = z^2$ or $3^x = z^2 - 1$.
Then $3^x = (z-1)(z+1)$.
Thus $(z-1) = 3^u$, where u is a non-negative integer .then $(z+1) = 3^{x-u}$

So that $2=3^{x-u}-3^{u}=3^{u}(3^{x-2u}-1)$.Now we have (i) $3^{u}=3^{0}$, Which implies that u=0 and $3^{x}-1=2$ i.e., $3^{x}=3$, which gives x=1, putting x=1 in $3^{x}+1=z^{2}$ we have $4=z^{2}$ i.e. z=2.

Hence (x, z)=(1,2) is a unique solution to the Diophantine equation $3^{x}+1=z^{2}$,where x, z are non-negative integers.

2.2 Lemma: The Diophantine equation $1+7^{y}=z^{2}$ has no solutions for all non-negative integers y, z.

Proof: Suppose that there are non-negative integers y and z such that $1+7^{y}=z^{2}$.

If y=0, then $z^{2}=2$ which is not possible. If z=0 then $7^{y}=-1$ which is impossible.

We consider the case when y, z>0. Then $z^{2}=1+7^{y}$ or $7^{y}=z^{2}-1=(z-1)(z+1)$.

Thus$(z-1)=7^{v}$, where v is a non-negative integer, then $(z+1)=7^{y-v}$

Thus $2=7^{y-v}-7^{v}=7^{v}(7^{y-2v}-1)$ which implies that v=0 and $7^{y}-1=2$ or $7^{y}=3$ which is not solvable. Hence the Diophantine equation $1+7^{y}=z^{2}$ has no solutions for all non-negative integers y, z.

III. MAIN RESULT

3.1 Theorem: The Diophantine equation $3^{x}+7^{y}=z^{2}$ has two solutions in non-negative integers (x, y, z) = (1, 0, 2) and (2, 1, 4)

Proof: let x, y and z be non-negative integers such that $3^{x}+7^{y}=z^{2}$ .we first consider the case when y is zero.

By lemma 2.1 we have **(x, y, z) = (1, 0, 2)** is a solution in non-negative integers x, y, z.

No solutions when x is zero by lemma 2.2.

From lemma 2.2,$x\geq 1$.Now we will divide y into two cases when$x\geq 1$.

Case (i) if y is even i.e. y=2k, for some positive integer k, then $3^{x}=z^{2}-7^{y}=z^{2}-7^{2k}=(z-7^{k})(z+7^{k})$ .Thus $z-7^{k}=3^{w}$ for some non-negative integer w. Then$3^{x}=3^{w}(z+7^{k})$

Or $z+7^{k}=3^{x-w}$.Now$2.7^{k}=3^{x-w}-3^{w}$ $=3^{w}(3^{x-2w}-1)$ .We have w =0 then $z-7^{k}=1$

Or $z=7^{k}+1$ which implies that the equation $z+7^{k}=3^{x}$ is not solvable until k=0.

So that (x, y, z) = (1, 0, 2).

Case (ii) when y is odd i.e. let y=2k+1, where k is a non-negative integer. We will divide this case into two parts i.e. part (i) and part (ii).

Part (i): $3^{x}+7^{2k+1}=z^{2}$ or $3^{x}+(3+4)7^{2k}=z^{2}$ i.e. $3^{x}+3.7^{2k}=z^{2}-4.7^{2k}$=$(z-2.7^{k})(z+2.7^{k})$

There are two possibilities for this equation

$$\begin{cases} z-2.7^{k}=1 \\ z+2.7^{k}=3^{x}+3.7^{2k} \end{cases} \quad \text{Or} \quad \begin{cases} z+2.7^{k}=1 \\ z-2.7^{k}=3^{x}+3.7^{2k} \end{cases}$$

Solving the first set of equations we get $2(2.7^{k})=3^{x}+(3.7^{2k})-1$

i.e.$3^{x}-1=(4.7^{k})-(3.7^{2k})=7^{k}\left(4-(3.7^{k})\right)$

which implies that k=0 and $3^{x}=2$,which is not solvable.

Solving the second set of equations, $2(2.7^{k})=1-3^{x}-(3.7^{2k})$ this gives

$1-3^{x}=4.7^{k}+(3.7^{2k})$

$=7^{k}(4+(3.7^{k}))$ which implies that k=0 and $1-3^{x}=7$ or $3^{x}=-6$ which is not solvable for x.

Part (ii) again we have, $3^{x}+7^{2k+1}=z^{2}$

Or $3^{x}+(36-29)7^{2k}=z^{2}$

i.e. $3^{x}+(-29)7^{2k}=z^{2}-(36)(7^{2k})$

$=(z-(6)(7^{k}))(z+(6)(7^{k}))$

There are two possibilities for this equation

$$\begin{cases} z-6.7^{k}=1 \\ z+6.7^{k}=3^{x}-(29.7^{2k}) \end{cases} \quad \text{Or} \quad \begin{cases} z-6.7^{k}=3^{x}-(29.7^{2k}) \\ z+6.7^{k}=1 \end{cases}$$

Solving the first set of equations, $3^{x}-1$ $=7^{k}(12+29.7^{k})$ This implies that k=0 and $3^{x}-1=(12+29)$=41or $3^{x}=42$

This is not solvable. Solving the second set of equations, $3^{x}-1=7^{k}(29.7^{k}-12)$ it implies that k=0 and $3^{x}-1=(29-12)$=17 or$3^{x}=18$, this is not solvable.

Similarly for x two cases

Case (i) when x is even i.e., x=2k, for some positive integer k, then $7^{y}=z^{2}-3^{2k}$=$(z-3^{k})(z+3^{k})$ thus $(z-3^{k})=7^{w}$ for some integer w>0 and $(z+3^{k})=7^{y-w}$ .Now solving we get

$2(3^{k})$= $7^{y-w}-7^{w}$=$7^{w}(7^{y-2w}-1)$.we have w=0 and $2(3^{k})$=$(7^{y}-1)$ and z=$3^{k}+1$ this implies that k=1 only so that $2(3^{1})$=$(7^{y}-1)$ i.e, $7^{y}=7$ $i.e,\ y=1$, z=3+1=4 and x=2. Therefore (x, y, z) = (2, 1, 4).

Case (ii) when x is odd i.e., x=2k+1 for some integer k>0, then $7^{y}=z^{2}-3^{2k+1}=z^{2}-(4-1)3^{2k}$

i.e. $7^{y}-3^{2k}=z^{2}-(4)3^{2k}$

=$(z-(2.3^{k}))(z+(2.3^{k}))$.

There are two possibilities for this equation

$$\begin{cases} (z-(2.3^{k}))=1 \\ (z+(2.3^{k}))=7^{y}-3^{2k} \end{cases} \quad \text{Or} \quad \begin{cases} (z-(2.3^{k}))=7^{y}-3^{2k} \\ (z+(2.3^{k}))=1 \end{cases}$$

Solving the first set of equations we get

$2(2.3^{k})=7^{y}-3^{2k}-1$ this implies that $7^{y}-1$ $=2(2.3^{k})+3^{2k}$=$3^{k}(4+3^{k})$ this implies that k=0 and $7^{y}-1=5$ i.e,$7^{y}=6$ not solvable.

Solving the second set of equations we get

$2(2.3^{k})=1-7^{y}+3^{2k}$ or $7^{y}-1=3^{2k}-2(2.3^{k})$ this implies that k=0 and$7^{y}=-2$, this is not solvable.

Hence (x, y, z) = (1, 0, 2) and (2, 1, 4) are the solutions of the Diophantine equation $3^x + 7^y = z^2$ for non-negative integers x, y and z.

3.2 Corollary: The Diophantine equation $3^x + 7^y = w^{2n}$ has two solutions (x, y, w, n) = (1, 0, 2, 1) and (2, 1, 4, 2) where x, y, w, n>0 are non-negative integers.

Proof: suppose that x, y and z non-negative integers such that $3^x + 7^y = w^4$.--(I)

Let $z = w^n$ then equation (I) becomes $3^x + 7^y = z^2$.then by theorem 3.1, we have (x, y, z) = (1, 0, 2) and (2, 1, 4).

When (x, y, z) = (1, 0, 2), we have $w^n = z = 2$ i.e $w= 2, n = 1$ (x, y, w, n) = (1, 0, 2, 1)

When (x, y, z) = (2, 1, 4), we have $w^n = z = 4$ i.e $w= 2, n = 2$ (x, y, w, n) = (2, 1, 2, 2). This completes the proof.

IV. OPEN PROBLEM

Let p and q be positive prime numbers. We may ask for the set of all solutions (x, y, z) for the Diophantine Equation $p^x + q^y = z^2$ where x, y and z are non-negative integers.

V. CONCLUSION

In this theorem, we have solved the Diophantine Equation $3^x + 7^y = z^2$, where p=3,q=7 are prime numbers and q-p=4. We have shown that the entitled equation has two solutions (x, y, z) = (1, 0, 2) and (2, 1, 4).

REFERENCES

1. *Z.Cao, A note on the Diophantine equation $a^x + b^y = c^z$,Acta Arith., XCI (1999), no. 1, 85-89.*
2. *B.Peker, S.Cenberci, Solutions of the Diophantine Equation $4^x + p^y = z^{2n}$, Seljuk J. Appl. Math. , 13 (2012), no. 2, 31-34.*
3. *B.Sroysang, More on the Diophantine Equation $8^x + 19^y = z^2$, Int. J. Pure. Appl. Maths., 81(2012), 601-604.*
4. *B.Sroysang,On the Diophantine Equation $8^x + 13^y = z^2$, Int. J. Pure. Appl. Maths., 90(2014), no.1, 69-72.https://doi.org/10.12732/ijpam.v90i1.9*
5. *B.Sroysang,on the Diophantine Equation $7^x + 8^y = z^2$, Int. J. Pure. Appl. Maths., 84(2013), 111-114.https://doi.org/10.12732/ijpam.v84i1.8*
6. *A.Suvarnamani, On a Diophantine Equation $2^x + p^y = z^2$,Int. J. Math. Sci. Appl., 1(2011), 1415-1419.*
7. *T.Hadano, On the Diophantine Equation $a^x = b^y + c^z$, Math. J. Okayama Univ., 19 (1976), 25-29.*
8. *J.F.T. Robago, On Two Diophantine Equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ Int. J. of Math. And Scientific Computing, 3(2013), 28-29*
9. *B. Sroysang, More on the Diophantine Equation $3^x + 5^y = z^2$, Int. J. Pure. Appl. Maths., 81(2012), 605-608.*
10. *P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math., 27 (2004), 167-195.*
11. *S.Chotchaisthit, On the Diophantine Equation $2^x + 11^y = z^2$, Maejo Int. J. Sci. Technol., 7(2013), 291-293.*
12. *A.Suvarnamani, On a Diophantine Equation $p^x + q^y = z^2$,Int. J. Of Pure Applied Mathematics, Volume 94 No. 4 (2014), 457-460... (Where q-p=2)*
13. *Shivangi Asthana, On the Diophantine Equation $8^x + 113^y = z^2$,Int. J. of Algebra, Vol. 11, 2017, no. 5, 225-230.*
14. *Ch. Gnanendra Rao, on the Ternary Quadratic Diophantine Equation $2x^2 - 3xy + 2y^2 = 56z^2$,IRJET, Vol-5, Issue-2, Feb-2018.*

AUTO-BIOGRAPHY

Chikkavarapu Gnanendra Rao,(born on 31-1-83 in Nuzvid, Andhra Pradesh) completed post-graduation in mathematics from P.B.Siddartha College of arts and science in Vijayawada in the year 2005. Presently he is working as assistant professor of mathematics in Sri Sarathi Institute of Engineering and Technology, Nuzvid. He was qualified in GATE 2013 and APSET-2012.