ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 6 | Issue: 12 |December 2020 || Journal DOI: 10.36713/epra2013 || SJIF Impact Factor: 7.032 ||ISI Value: 1.188

TWO CURIOUS DEFINITE INTEGRAL

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ABSTRACT
In this paper we have established two definite integral in the form of hypergeometric function.
KEY WORDS : *Elliptic Integral,Gauss hypergeometric function, Boolean Algebra.*
2020 MSC NO: 33C75,33E05,03G05

1. INTRODUCTION

Yurry A. Brychkov[Brychkov p.155-156(4.1.6)] has established the following formulae

$$\int_0^a \frac{1}{\sqrt{(a^2-x^2)}}\, sin^{-1}(bx)\, dx = \frac{1}{2}[Li_2(ab)+Li_2(-ab)],\ \ [|\arg(1-a^2b^2)|<\pi]. \quad (1.1)$$

$$\int_0^a \sqrt{(a^2-x^2)}\, sin^{-1}(bx)\, dx = \frac{a}{4b}\{\frac{1-a^2b^2}{2ab}\ln\frac{1+ab}{1-ab} + ab[Li_2(ab)-Li_2(-ab)]-1\},\ \ [|\arg(1-a^2b^2)|<\pi]. \quad (1.2)$$

$$\int_0^a x\sqrt{(a^2-x^2)}\, sin^{-1}(bx)\, dx = \frac{a^2}{9b}[2(1-2a^2b^2)D(ab)-(1-3a^2b^2)K(ab)],\ \ [|\arg(1-a^2b^2)|<\pi]. \quad (1.3)$$

$$\int_0^a \frac{x}{\sqrt{(a^2-x^2)}}\, sin^{-1}(bx)\, dx = a^2b[K(ab)-D(ab)],\ \ [|\arg(1-a^2b^2)|<\pi]. \quad (1.4)$$

A generalized hypergeometric function ${}_\alpha F_\beta(a_1,\ldots,a_\alpha;b_1,\ldots,b_\beta;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{\zeta+1}}{c_\zeta} = \frac{P(\zeta)}{Q(\zeta)} = \frac{(\zeta+a_1)(\zeta+a_2)\ldots(\zeta+a_\alpha)}{(\zeta+b_1)(\zeta+b_2)\ldots(\zeta+b_\beta)(\zeta+1)}\, z. \quad (1.5)$$

Where $\zeta+1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

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$$ {}_{\alpha}F_{\beta}\left[\begin{matrix} a_1, a_2, \cdots, a_\alpha & ; & \\ & & z \\ b_1, b_2, \cdots, b_\beta & ; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_\alpha)_k z^k}{(b_1)_k (b_2)_k \cdots (b_\beta)_k k!} \tag{1.6} $$

where the parameters $b_1, b_2, \cdots, b_\beta$ are positive integers.
The complete elliptic integral of the first kind K is defined as

$$ K(\eta) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\eta^2 \sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\eta^2 t^2)}}, \tag{1.7} $$

In power series

$$ K(\eta) = \frac{\pi}{2} \sum_{n=0}^{\infty} [\frac{(2n)!}{2^{2n}(n!)^2}]^2 \eta^{2n} = \frac{\pi}{2} \sum_{n=0}^{\infty} [P_{2n}(0)]^2 \eta^{2n}, \tag{1.8} $$

where P_n is the Legendre polynomial.
In terms of the Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$ K(\eta) = \frac{\pi}{2} \quad {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \eta^2) \tag{1.9} $$

The complete elliptic integral of the second kind E is defined as

$$ E(\eta) = \int_0^{\frac{\pi}{2}} \sqrt{1-\eta^2 \sin^2\theta}\ d\theta = \int_0^1 \frac{\sqrt{1-\eta^2 t^2}}{\sqrt{1-t^2}}\ dt. \tag{1.10} $$

It can be expressed as a power series

$$ E(\eta) = \frac{\pi}{2} \sum_{n=0}^{\infty} [\frac{(2n)!}{2^{2n}(n!)^2}]^2 \frac{\eta^{2n}}{1-2n}. \tag{1.11} $$

In terms of the Gauss hypergeometric function

$$ E(\eta) = \frac{\pi}{2} \quad {}_2F_1(\frac{1}{2}, -\frac{1}{2}; 1; \eta^2). \tag{1.12} $$

The fundamental operations of Boolean algebra are as follows:
AND (conjunction), denoted $\xi \wedge \omega$, satisfies $\xi \wedge \omega = 1$ if $\xi = \omega = 1$, and $\xi \wedge \omega = 0$ otherwise.
OR (disjunction), denoted $\xi \vee \omega$, satisfies $\xi \vee \omega = 0$ if $\xi = \omega = 0$, and $\xi \vee \omega = 1$ otherwise.
NOT (negation), denoted $\neg\xi$, satisfies $\neg\xi = 0$ if $\xi = 1$ and $\neg\xi = 1$ if $\xi = 0$.
The dilogarithm $Li_2(z)$ is a special case of the polylogarithm $Li_n(z)$ for $n = 2$.
The dilogarithm can be defined as

$$ Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}. \tag{1.13} $$

Main Formulae of the Integration

$$\int_0^a \frac{x\ \cos^{-1}(cx)}{\sqrt{a^2-x^2}}\,dx = -\frac{1}{4}\pi\ a[ac\quad {}_2F_1(\frac{1}{2},\frac{1}{2};2;a^2c^2)-2]\ for\ c\in R\wedge Re(a) > 0\wedge Im(a)=0. \tag{2.1}$$

$$\int_0^a \frac{x^3\ \cos^{-1}(cx)}{\sqrt{a^2-x^2}}\,dx = \frac{1}{48}\pi\ a^3[16-9ac\quad {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{5}{2};\frac{3}{2},3;a^2c^2)]\ for\ c\in R\wedge Re(a) > 0\wedge Im(a)=0. \tag{2.2}$$

Derivation of Main Formulae

Derivation of (2.1)

$$\int_0^a \frac{x\ \cos^{-1}(cx)}{\sqrt{a^2-x^2}}\,dx = -[\frac{\sqrt{a^2-x^2}\{E(\sin^{-1}(cx)|\frac{1}{a^2c^2})+\sqrt{1-\frac{x^2}{a^2}}\cos^{-1}(cx)\}}{\sqrt{1-\frac{x^2}{a^2}}}]_0^a$$

$$= -\frac{1}{4}\pi\ a[ac\quad {}_2F_1(\frac{1}{2},\frac{1}{2};2;a^2c^2)-2]\ for\ c\in R\wedge Re(a) > 0\wedge Im(a)=0$$

Derivation of (2.2)

$$\int_0^a \frac{x^3\ \cos^{-1}(cx)}{\sqrt{a^2-x^2}}\,dx$$

$$= [\frac{1}{9\sqrt{-\frac{1}{a^2}}\ c^3\sqrt{a^2-x^2}}\{-\sqrt{-\frac{1}{a^2}}\ c^2(a^2-x^2)(3c(2a^2+x^2)\cos^{-1}(cx) - x\sqrt{1-c^2x^2})+$$

$$+\iota(5a^2c^2+2)\sqrt{1-\frac{x^2}{a^2}}\ E(\iota\sinh^{-1}(\sqrt{-\frac{1}{a^2}}\ x)|a^2c^2)+$$

$$+2\iota(3a^4c^4-2a^2c^2-1)\sqrt{1-\frac{x^2}{a^2}}\ F(\iota\sinh^{-1}(\sqrt{-\frac{1}{a^2}}\ x)|a^2c^2)\}]_0^a$$

$$= \frac{1}{48}\pi\ a^3[16-9ac\quad {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{5}{2};\frac{3}{2},3;a^2c^2)]\ for\ c\in R\wedge Re(a) > 0\wedge Im(a)=0$$

REFERENCES

1. *Abramowitz, Milton. A and Stegun, Irene ; Handbook of Mathematical Functions with Formulas , Graphs , and Mathematical Tables, National Bureau of Standards, 1970.*
2. *Aomoto, Kazuhiko, Kita, Michitake ; Theory of Hypergeometric functions, Springer Tokyo Dordrecht Heidelberg London New York, 2011.*
3. *Brychkov,Y.A.; Handbook of Special Functions: Derivatives,Integrals,Series and Other Formulas. CRC Press,Taylor & Francis Group, London, U.K, 2008.*
4. *Byrd, P. F., Friedman, M.D.; Handbook of Elliptic Integrals for Engineers and Scientists (2nd ed.), New York: Springer-Verlag,1971,ISBN 0-387-05318-2.*

5. *Erdélyi, Arthur, Magnus, Wilhelm, Oberhettinger, Fritz, Tricomi, Francesco G; Higher transcendental functions. Vol II, McGraw-Hill Book Company, Inc., New York-Toronto-London,1953.*
6. *Greenhill, Alfred George; The applications of elliptic functions, New York: Macmillan,1892.*
7. *Gauss, C. F.; Disquisitiones generales circa seriem infinitam ... , Comm. soc. reg. sci. Gott. rec., 2(1813), 123-162.*
8. *Koepf, W.; Hypergeometric Summation: An Algorithmic Approach to Summation and Special Function Identities. Braunschweig, Germany: Vieweg, 1998.*
9. *Luke, Y. L.; Mathematical functions and their approximations. Academic Press Inc., London,, 1975.*
10. *P. Appell, Sur une formule de M. Tisserand et sur les fonctions hypergéométriques de deux variables, J. Math. Pures Appl., (3) 10 (1884) 407-428.*
11. *Prudnikov, A.P., Brychkov, Yu. A. and Marichev, O.I.; Integral and Series Vol 3: More Special Functions,Nauka, Moscow,2003.*
12. *Salahuddin, Husain, Intazar; Certain New Formulae Involving Modified Bessel Function of First Kind, Global Journal of Science Frontier Research (F),13(10)(2013),13-19.*
13. *Salahuddin, Khola, R. K.;New hypergeometric summation formulae arising from the summation formulae of Prudnikov, South Asian Journal of Mathematics,4(2014),192-196.*
14. *Steffensen, J. F.; Interpolation (2nd ed.),Dover Publications,U.S.A,2006.*