CONNECTED COMPLEMENTARY TREE DOMINATION NUMBER OF GRAPHS

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ABSTRACT

Let $G = (V, E)$ be a non-trivial, simple, finite and undirected graph. A dominating set $D$ is called a complementary tree dominating set if the induced subgraph $<V-D>$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of $G$ and is denoted by $\gamma_{ctd}(G)$. A dominating set $D$ is called a connected complementary tree dominating set (cctd-set) if the induced subgraph $<D>$ is connected. The connected complementary tree dominating number $\gamma_{cctd}(G)$ of a connected graph $G$ is the minimum cardinality of a connected complementary tree dominating set of $G$.

In this paper, connected complementary tree dominating set, connected complementary number are defined and minimal connected complementary tree dominating set are characterized and bounds also obtained.

KEYWORDS: Connected domination, connected complementary tree domination.

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1. INTRODUCTION

The graph considered here are nontrivial, simple, finite and undirected. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$ the neighbourhood $N(v)$ of $v$ is the set of all vertices adjacent to $v$ in $G$. $N[v] = N(v) \cap \{v\}$ is called the closed neighborhood of $v$. The concept of domination was first studied by Ore [6]. A set $D \subseteq V$ is said to be a dominating set of $G$. If every vertex in $V-D$ is adjacent to some vertex in $D$. The minimum cardinality of a dominating set is called the domination number of $G$ and is denoted by $\gamma(G)$. The concept of complementary tree domination was introduced by S. Muthammai, M. Bhanumathi and P. Vidhya in [4]. A dominating set $D \subseteq V$ is called a complementary tree dominating (ctd) set, if the induced subgraph $<V-D>$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of $G$ and is denoted by $\gamma_{ctd}(G)$. The concept of connected domination in graphs was introduced by E. Sampathkumar and Walikar [5]. A dominating set $D \subseteq V$ is called a connected dominating set of $G = (V, E)$ if the induced subgraph $<V-D>$ is connected. The connected domination number $\gamma_c(G)$ of a connected graph $G$ is the minimum cardinality of a connected dominating set. A dominating set $D$ is called a connected complementary tree dominating set if the induced subgraph $<D>$ is connected. The connected complementary tree domination number $\gamma_{cctd}(G)$ of a connected graph $G$ is the minimum cardinality of a connected complementary tree dominating set. In this paper, connected complementary tree dominating set, connected complementary number are defined and minimum connected complementary tree dominating set and its bounds are obtained.
2. RESULTS

Theorem 2.1. [3]

If $G$ is a connected graph with $p \geq 3$ vertices, then $\gamma_c(G) = p - \varepsilon_t(G)$ where $\varepsilon_t(G)$ is the maximum number of pendant edges in any spanning tree of $G$.

Theorem 2.2. [5]

For any connected graph $G$, \[
\frac{p}{\Delta(G) + 1} \leq \gamma_c(G) \leq 2q - p.
\]

Theorem 2.3. [2]

For any connected graph $G$, $\gamma_c(G) \leq p - \Delta(G)$.

3. CONNECTED COMPLEMENTARY TREE DOMINATION NUMBER OF GRAPHS

In this section, connected complementary tree dominating set, connected complementary tree domination number are defined and minimal connected complementary tree dominating sets are characterized. Also bounds of connected complementary tree domination number are obtained.

In the following, connected complementary tree domination number is defined.

Definition 3.1:

A complementary tree dominating set $D \subseteq V$ of a connected graph $G = (V, E)$ is said to be a connected complementary tree dominating set (cctd-set), if the induced subgraph $<D>$ is connected.

The connected complementary tree domination number $\gamma_{cctd}(G)$ of a connected graph $G$ is the minimum cardinality of a connected complementary tree dominating set of $G$.

A $\gamma_{cctd}$-set is a minimum connected complementary tree dominating set.

A connected complementary tree dominating set is said to be minimal, if no proper subset of $D$ is connected complementary tree dominating set.

It is to be noted that $\gamma_{cctd}$-set exists, for all connected graphs.

Example 3.1.

Consider the graph in Figure 1.

![Figure 1](image-url)

The sets $D_1 = \{v_2, v_3\}$ and $D_2 = \{v_3, v_4\}$, $D_3 = \{v_2, v_3\}$, $D_4 = \{v_5, v_6\}$ are connected complementary tree dominating sets of minimum cardinality.

Therefore, $\gamma_{cctd}(G) = 2$.

Observation 3.1.

(i) Let $G$ be a connected graph with $\Delta(G) < p-1$.

Since, every cctd-set is a connected tree dominating set,

$\gamma_{cctd}(G) \leq \gamma_{cctd}(G)$ for any connected graph $G$.

Also, every cctd-set is a connected dominating set.

Therefore $\gamma_c(G) \leq \gamma_{cctd}(G)$ for any connected graph $G$.

Equality holds, if $G$ is the complete bipartite graph $K_{2n}$ ($n \geq 2$).

(ii) Let $H$ be a connected spanning subgraph of a connected graph $G$. It is not necessary that the inequality $\gamma_{cctd}(G) \leq \gamma_{cctd}(H)$ holds. For example, $\gamma_{cctd}(K_3) = 3$ and $\gamma_{cctd}(K_3 - e) = 2$. 
Let \( G \) be the subdivision graph of star \( K_{1,n} \). Then \( \gamma_{ctd}(G) = 2n, n \geq 2 \).

\[ \gamma_{ctd}(C_n^0) = (n-1)t - 1. \]

Here, \( C_n^0 \) has \( p = (n-1)t + 1 \) vertices. Here, the point of union of cycles and all the vertices of the cycles except two vertices from any one of the cycle forms a cctd-set of \( G \). Therefore, \( \gamma_{ctd}(C_n^0) = p - 2 = (n-1)t - 1 \).

**Theorem 3.2.**

Let \( G_1 \) and \( G_2 \) be any two connected graphs of order at least 3. Then, \( \gamma_{ctd}(G_1 \circ G_2) \leq |V(G_1)| \cdot (1 + |V(G_2)|) - |V(T)| \), where \( T \) is a subgraph of \( G_2 \) which is a tree with maximum number of vertices.

**Proof.**

The set \( V(G_1 \circ G_2) - V(T) \) is a cctd-set of \( G_1 \circ G_2 \) and hence

\[ \gamma_{ctd}(G) \leq |V(G_1 \circ G_2)| - |V(T)| = |V(G_1)| \cdot (1 + |V(G_2)|) - |V(T)| \]

Equality holds, if \( G_1 \cong C_3 \) and \( G_2 \cong C_3 \).

**Corollary 3.1.**

If \( G_2 \) is a tree, then

\[ \gamma_{ctd}(G_1 \circ G_2) = |V(G_1)| \cdot (1 + |V(G_2)|) - (|V(G_2)| - 1) \]

**4. BOUNDS AND EXACT VALUES FOR THE CONNECTED COMPLEMENTARY TREE DOMINATION NUMBER**

**Observation 4.1.**

For any connected graph \( G \), \( \gamma_{ctd}(G) \geq \left[ \frac{p}{\Delta(G) + 1} \right] \).

Since \( \gamma_{ctd}(G) \geq \gamma_1(G) \) and \( \gamma_1(G) \geq \left[ \frac{p}{\Delta(G) + 1} \right] \),

the inequality \( \gamma_{ctd}(G) \geq \left[ \frac{p}{\Delta(G) + 1} \right] \) holds.

This bound is attained, if \( G \cong C_\ell \) and \( C_5 \).

**Theorem 4.1.**

For any connected \((p, q)\) graph \( G \), \( \gamma_{ctd}(G) \geq 2p - q - 2 \).
Proof.

Let D be a \( \gamma_{\text{null}} \)-set of G since \( <D> \) is connected, number of edges in \( <D> \) is greater than or equal to \( |D| - 1 = \gamma_{\text{null}}(G) - 1 \).

Number of edges in \( <V-D> \) is \( p - \gamma_{\text{null}}(G) - 1 \).

There are at least \( p - \gamma_{\text{null}}(G) \) edges from \( V - D \) to \( D \).

Therefore, \( q \geq \gamma_{\text{null}}(G) - 1 + p - \gamma_{\text{null}}(G) + p - \gamma_{\text{null}}(G) - 1 \)

That is, \( q \geq 2p - \gamma_{\text{null}}(G) - 2 \).

Hence, \( \gamma_{\text{null}}(G) \geq 2p - q - 2 \).

Equality holds, if \( G \cong K_{p+1} \), \( n \geq 3 \). □

**Theorem 4.2.**

Let \( G \) be a connected graph with at least three vertices and let \( T \) be a spanning tree of \( G \) with maximum number \( \varepsilon_{\text{ctd}}(G) \) of pendant vertices. Then, \( \gamma_{\text{null}}(G) = p - \varepsilon_{\text{ctd}}(G) \), if and only if the subgraph of \( G \) induced by pendant vertices of \( T \) is a tree in \( G \).

**Proof.**

Let \( S \) be the set of all pendant vertices in \( T \). Therefore, \( |S| = \varepsilon_{\text{ctd}}(G) \). Then \( V - S \) is a connected dominating set.

Assume \( <S> \) is a tree. Then, \( V - S \) is an \( \varepsilon_{\text{ctd}} \)-set of \( G \) and hence,

\( \gamma_{\text{null}}(G) \leq |V - S| = p - \varepsilon_{\text{ctd}}(G) \). But, \( \gamma_{\text{null}}(G) \geq \varepsilon_{\text{ctd}}(G) = p - \varepsilon_{\text{ctd}}(G) \).

Therefore, \( \gamma_{\text{null}}(G) = p - \varepsilon_{\text{ctd}}(G) \).

Conversely, assume \( \gamma_{\text{null}}(G) = p - \varepsilon_{\text{ctd}}(G) \). Then, there exists an \( \varepsilon_{\text{ctd}} \)-set \( D \) such that \( |D| = p - \varepsilon_{\text{ctd}}(G) \).

If the subgraph \( <S> \) of \( G \) is not a tree, then \( V - S \) will not be a \( \varepsilon_{\text{ctd}} \)-set.

Hence, \( <S> \) is a tree in \( G \). □

**Theorem 4.3.**

Let \( G \) be a connected graph such that \( \text{diam}(G) = 2 \). If the subgraph of \( G \) induced by neighbourhood set of a vertex of maximum degree is a tree in \( G \), then

\( \gamma_{\text{null}}(G) \leq p - \Delta(G) + 1 \).

**Proof.**

Let \( v \) be a vertex of maximum degree in \( G \) such that \( <N(v)> \) is a tree. Let \( u \) be a pendant vertex in \( <N(v)> \), then \( (V - N(v)) \cup \{u\} \) is a \( \varepsilon_{\text{ctd}} \)-set of \( G \). Hence,

\( \gamma_{\text{null}}(G) \leq |V - N(v)| \cup \{v\} \) and hence, \( \gamma_{\text{null}}(G) \leq p - \Delta(G) + 1 \). □

**Theorem 4.4.**

Let \( G \) be a connected graph which is not complete such that \( \gamma(G) = 1 \) and \( \delta(G) \geq 2 \). Then,

\( \gamma_{\text{null}}(G) \leq p - 3 \).

**Proof.**

Since \( G \) is not complete, there exists at least one pair of nonadjacent vertices. Hence, there exists an induced path \( P_3 \) on three vertices.

Then, \( V(G) - V(P_3) \) is a \( \varepsilon_{\text{ctd}} \)-set of \( G \) and hence \( \gamma_{\text{null}}(G) \leq p - 3 \).

This bound is attained, if \( G \cong K_{n-1} - e, n \geq 4 \). □

**Theorem 4.5.**

Let \( G \) be any nontrivial connected graph of order at least three.

Then, \( \gamma_{\text{null}}(G) = 1 \) if and only if \( G \cong T + K_1 \), where \( T \) is a tree.

**Proof.**

The proof is in similar lines to Proposition 3.6 [4].

Next, the graphs \( G \) for which \( \gamma_{\text{null}}(G) = 2 \) are found.

**Theorem 4.6.**

Let \( G \) be a connected graph with at least 4 vertices. Then, \( \gamma_{\text{null}}(G) = 2 \) if and only if \( G \) is one of the following graphs

(i) \( G \) is the graph \( K_1 + T \) with one pendant edge attached at the vertex of \( K_1 \), where \( T \) is a tree.

(ii) \( G \) is the graph obtained from a tree by joining each of the vertices of the tree to the vertices of \( K_2 \) such that \( \text{deg}_G(v) \geq 2 \), for all \( v \in V(K_2) \).

**Proof.**

Let \( G \) be one of the graph mentioned in (i) and (ii). Since \( G \) is not isomorphic to \( K_1 + T \). For any tree \( T, \gamma_{\text{null}}(G) \geq 2 \).

If \( G \) is the graph as in (i), the subset of \( V(G) \) consisting of the vertex of \( K_1 \) and the pendant vertex of \( G \) forms a \( \varepsilon_{\text{ctd}} \)-set of \( G \).

Therefore, \( \gamma_{\text{null}}(G) \leq 2 \) and hence \( \gamma_{\text{null}}(G) = 2 \). Conversely, assume \( \gamma_{\text{null}}(G) = 2 \) then, there exists a \( \varepsilon_{\text{ctd}} \)-set \( D \) such that \( |D| = 2 \).

Let \( D = \{u, v\} \).
a) If u or v is a pendant vertex in G, then all the vertices of V−D are adjacent to v or u. Therefore, G is the graph mentioned in (i).
b) Let \( \text{deg}_G(u) \geq 2 \) and \( \text{deg}_G(v) \geq 2 \). Since \(<V-D>\) is a tree and D is a tree and D is a connected dominating set of G, each vertex in V−D is adjacent to at least one vertex in D. Hence, G is the graph as in (ii).

\[ \square \]

**Theorem 4.7.**

Let G be a connected graph. Then, \( \gamma_{cctd}(G) = p-1 \) if and only if either

(i) the subgraph of G induced by vertices of G which are not the cutvertices is totally disconnected (or) contains one vertex of G or

(ii) each vertex of G of degree at least 2 is a cutvertex.

**Proof.**

Let either the subgraph of G induced by vertices which are not the cutvertices of G be totally disconnected (or) each vertex in G of degree at least 2 is a cutvertex. Then, \( \delta(G) = 1 \). Let D be a cctd-set of G. Since the set \( F \subseteq V(G) \) of cutvertices forms a connected dominating set, \( F \subseteq D \) and also \(<V-F>\) is a totally disconnected.

Since \(<V-D>\) is a tree, all the vertices of V−F except one vertex must belong to D and hence, \( |D| \geq p-1 \). But, \( |D| \leq p-1 \) and hence, \( |D| = p-1 \).

Conversely, assume \( \gamma_{cctd}(G) = p-1 \).

Then, there exists a cctd-set D of G such that \( |D| = p-1 \).

Let \(<V-D> = \{v\}\). Since D is a dominating set, v is adjacent to at least one vertex in D. If the subgraph H of G induced by vertices which are not the cutvertices is not totally disconnected (or) contains at least two vertices, then there exists at least one edge \((u, v)\) in H. Then, the set \( V - \{u, v\} \) will be a cctd-set of G and hence, \( \gamma_{cctd}(G) \leq p-2 \), which is a contradiction. Hence, the theorem is proved. \( \square \)

**Corollary 4.1.**

If G is a tree with p vertices, then \( \gamma_{cctd}(G) = p-1 \).

**Theorem 4.8.**

Let \( p \geq 4 \) be an integer. For each \( k \) satisfying \( 2 \leq k \leq p-2 \), there is a connected graph \( G \) such that \( \gamma_{cctd}(G) = \gamma_{cctd}(G') = k \).

**Proof.**

Construct a graph \( G \) as follows. Attach \( k-2 \) pendant edges at exactly one vertex of \( K_4 \) or \( C_3 \) (\( 2 \leq k \leq p-2 \)).

For this graph, \( \gamma_{cctd}(G) = \gamma_{cctd}(G') = k \).

\( \square \)

**Theorem 4.9.**

Let G be a connected graph with at least three vertices. Then, \( \gamma_{cctd}(G) = p-2 \) if and only if

(i) \( G \cong K_p \) (or) \( C_p \)

(ii) If S is the set of all cutvertices of G such that \(<V-S>\) is not totally disconnected, then at least one of the following holds.

(a) components of \(<V-S>\) are complete graphs \( K_n \) \( n \geq 1 \), \( n \neq 2 \).

(b) If there exists an induced path \( P \) of length 2 in a component of \(<V-S>\), then the central vertex of \( P \) is of degree 2 in G.

**Proof.**

Assume \( \gamma_{cctd}(G) = p-2 \). Then, there exists a \( \gamma_{cctd} \) set D such that \( |D| = \gamma_{cctd}(G) \) and \(<V-D> \cong K_p \).

Let S be the set of all cutvertices of G such that \(<V-S>\) is not totally disconnected.

**Case 1.** \( S = \emptyset \).

If there exists an induced path \( P \) of length 2 such that the central vertex of \( P \) is of degree at least 2, then \( V(G) - V(P) \) is a cctd-set of \( G \) and hence \( \gamma_{cctd}(G) \leq p-3 \).

Therefore, one of the following holds.

(i) There exists no induced path of length 2 in G.

(ii) Central vertex of every induced path is of degree 2 in G.

If (i) holds, then \( G \cong K_p \).

If (ii) holds, then \( G \cong C_p \).

**Case 2.** \( S \neq \emptyset \).

Then \( S \subseteq D \).

By the same argument above, if there exists an induced path \( P \) of length 2 in a component of \(<V-S>\) and if the central vertex has degree at least 3 in G, then

\( \gamma_{cctd}(G) \leq p-3 \). Hence, at least one of (a) and (b) of (ii) holds.

Conversely, if \( G \cong C_p \), then \( \gamma_{cctd}(G) = p-2 \).

If at least one of (a) and (b) of (ii) holds, then there exists a minimum cctd-set of \( G \) consisting of \( (p-2) \) vertices and hence, \( \gamma_{cctd}(G) = p-2 \). \( \square \)
Theorem 4.10.

Let G be a graph such that G and its complement $\overline{G}$ are connected. Then

$$4 \leq \gamma_{cctd}(G) + \gamma_{cctd}(\overline{G}) \leq 2(p - 1)$$

$$4 \leq \gamma_{cctd}(G) + \gamma_{cctd}(\overline{G}) \leq (p - 1)^2$$

REFERENCES