ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 6 | Issue: 12 |December 2020 || Journal DOI: 10.36713/epra2013 || SJIF Impact Factor: 7.032 ||ISI Value: 1.188

SARD'S THEOREM AND ITS APPLICATIONS

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ABSTRACT

The aim of this paper is to provide an introduction to Regular Value Theorem and Sard's Theorem on manifolds and some applications. A critical point of a smooth map is a point of the domain at which the derivative does not have full rank. Our goal, then, is to discuss critical points of smooth maps and some applications on topology.

First of all, we will define the terms and prove some of their basic proper-ties, in order to provide the necessary background to help reader understand the content we will talk about later. By smooth, we mean differentiable to all orders. By manifold, we mean a subset of Euclidean space. We will also illustrate examples and discuss diffeomorphisms.

1. CONCEPT OF SMOOTH MAPS

$\mathbb{R}^n$ denotes the n-dimensional euclidean space, and for x ∈ $\mathbb{R}^n$, we obtain

x = (x_1, x_2,…, x_n), such that x_1,x_2,…, x_n∈ $\mathbb{R}^n$which is the real numbers.

Then we will define what is a smooth map.

Definition :1.1.

Let U ⊂ $\mathbb{R}^n$ and V ⊂ $\mathbb{R}^k$ be open sets. A mapping f fromU to V written as

f : U → V is a smooth map if all of the partial derivatives exist and are continuous.

More generally, we have the following.

Definition :1.2.

Let X ⊂ $\mathbb{R}^n$ and Y ⊂ $\mathbb{R}^k$ be arbitrary subsets, and a map f : X → Y is called smooth map if for every x ∈ X there exist an open set O ⊂ $\mathbb{R}^n$ that x ∈ O and a smooth mapping F : O → $\mathbb{R}^k$ that coincides with f throughout O ∩ X.

Definition :1.3.

Let U ⊂ $\mathbb{R}^n$ and V ⊂ $\mathbb{R}^n$ be open sets. For any smooth map f: U → V , the derivative df_x : $\mathbb{R}^n$ → $\mathbb{R}^k$ is defined by the formula

$$df_x(h) = \lim_{t\to 0}(f(x+th) - f(x))/t$$

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for $x \in U$, $h \in \mathbb{R}^n$. Clearly $df_x(h)$ is a linear function of h.

Definition :1.4.

A map : U → V is called a diffeomorphism if f carries X onto Y and also both f and f^{-1} are smooth. Moreover, a smooth map f : X → Y of subsets of two euclidean spaces is a diffeomorphism if it is one-to-one and onto, and if the inverse map f^{-1} : Y → X is also a smooth map.

Note that diffeomorphisms are between two manifolds of the same dimension only.

Definition :1.5:

A diffeomorphism $\emptyset$: U → V is called a parametrization of the neighborhood V

Consider some examples.

Example :1.6. Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^l$, and let f : X → Y and g : Y → Z be smooth maps. If f and g are diffeomorphism, so is $g o f$.

Proof. If f and g are diffeomorphisms, then $g o f$ is bijective and smooth, being a composition of smooth maps. Moreover, we have that

$(gof)^{-1} = f^{-1} o g^{-1}$

is also a composition of smooth maps, and therefore is smooth. Thus gof is also be a diffeomorphism.

The Inverse Function Theorem will provide the keys to understanding the local structure of a smooth map.

Theorem :1.7 (Inverse Function Theorem) Let U, V be open sets of $\mathbb{R}^n$.If

f : U → V is a smooth map and at a point p the jacobian matrix d f_p is invertible, then there is a neighborhood U of p on which : U → f(U) is a diffeomorphism.

Definition :1.8

A subset $M \subset \mathbb{R}^k$ is called a smooth manifold of dimension k if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset $V \subset \mathbb{R}^k$.

Example :1.9 The following are examples of smooth manifolds:

- Open subsets of smooth manifolds.
- The set of all matrices. Moreover, the set of invertible matrices, since it is an open subset of the set of all matrices.

- Products of smooth manifolds.

2- REGULAR VALUE

As we will introduce Sard's theorem later, we need to give definitions of some important terms that we will use in Sard's theorem in order to make the theorem will be easier to follow and understand.

Definition :2.1.

Let : U → V be a smooth map between same dimensional manifolds. We denote that x ∈ U is a regular point if the derivative is nonsingular.

Definition :2.2

For another case, if the derivative is singular, then x is called a critical point. We also say y ∈ V is a critical value if y is not a regular value.

Definition :2.3

Let : U → V be a smooth map between same dimensional manifolds. For y ∈ V is called a regular value if $f^{-1}(y)$ contains only regular points. Moreover, we can have that y ∈ V is called a regular value if

$$df_x : T_x(U) \rightarrow Ty(V)$$

is onto at x such that f(x) = y.

Note that the image of the set of critical points is the set of critical values. The set of regular values is the subset of image of the set of regular points.

Theorem :2.4 (Pre image Theorem) If y is a regular value of : X → Y then the pre image $f^{-1}(y)$ is a sub manifold of X with dim

$f^{-1}(y)$ = dim X - dim Y .

Theorem :2.5 (Implicit Function Theorem) If $f_1,\dots,f_m$ are smooth maps on $\mathbb{R}^n$ and the matrix of partial derivatives has rank m for all points p on the zero set

$$Z = \{x \in \mathbb{R}^n | f_1(x) = 0, \dots, f_m(x) = 0\}$$

then Z is a smooth manifold of dimension n - m.

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3- MEASURE ZERO

Since we also need to know about measure zero sets for Sard's theorem, we will give some definitions and examples. We say that a set A ⊂ $\mathbb{R}^k$ is measure zero if it can be covered by a countable number of rectangulars that have small total volume.

Definition :3.1

A subset A ⊂ $\mathbb{R}^k$is called measure zero if for any ∈> 0,there exists a countable union of open sets U_i∈ $\mathbb{R}^k$ such that $A \subset \bigcup_i u_i$ and total volume is small such that

$$\sum_i volume\ u_i < \in$$

Theorems:3.2

In this section, we will introduce some theorems that we will use in the future chapters. For proving the Sard's theorem, we need to know Fubini theorem and Taylor theorem, and also in order to show that S^2 is simply connected, we need to know Weierstrass approximation theorem and the Van Kampen theorem.

For the proof of Sard's theorem, we will need the measure zero form of Fubini's theorem. Suppose that n = k + l and $\mathbb{R}^n$ = $\mathbb{R}^k$x $\mathbb{R}^l$. For each c ∈ $\mathbb{R}^k$, let V_c be the vertical slice {c}x$\mathbb{R}^l$.

Theorem :3.3 (Fubini theorem) Let A be a closed subset of $\mathbb{R}^n$ such that A ∩ V_c has measure zero in V_c for all c ∈ $\mathbb{R}^k$. Then A has measure zero in $\mathbb{R}^n$.

Proof. Since A may be written as a countable union of compacts. Assume A is compact, and by induction on k, it suffices to prove the theorem for k = 1 and l = n - 1. Since A is compact, and we have an interval I = [a b] such that A ⊂V_I . For each c ∈ I, choose a covering of A ∩ V_c by n – 1 dimension a_i rectangular solids S_1(c), … , S_{Nc}(c) having a total volume less than ∈. Choose an interval J(c) in R so that the rectangular solids J(c) x S_i(c) cover A∩ V_J . The J(c)'s cover [ab], and therefore we replace them with a finite collection of subintervals $J'_j$with total length less than 2(b - a).

Thus each $J'_j$is contained in some interval J(c_j), so the solids J'_j x S_i(c_j) cover A, and also they have total volume less than 2∈(b - a).

Theorem :3.4 (Taylor Theorem) Assume that f is n + 1 times differentiable, and P_n is the degree n Taylor approximation of f with center c. Then if x is any other value, there exists some value b between c and x such that

$$f(x) = p_n(x) + \frac{f^{(n+1)}(b)}{(n+1)!}(x-c)^{n+1}$$

$$where\ p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^n(c)}{n!}(x-c)^n$$

Theorem :3.5 (Weierstrass approximation theorem) Let f ∈ C[a b]. Given any ∈> 0 there exists a polynomial p_n of sufficiently high degree n for which

$$| f(x)- p_n(x) |\leq \in$$

for all x ∈ [a b].

Theorem :3.6 (Van Kampen Theorem) If X = $U_1 \cup U_2$ with U_i open and path-connected, and U1 ∩U2 path-connected and simply connected, then the induced homomorphism

$$\emptyset : \pi_1(U_1) x \pi_1(U_2) \to \pi_1(X)$$

is an isomorphism.

Immersions and Submersions:

Using the differential, we classify maps having maximal rank at a point into immersions and submersions at the point, depending on whether the differential is injective or surjective there.

4- IMMERSIONS

Definition :4.1.

If : X → Y carries x to y, and its derivative d f_x :$T_x(X) \to T_y(Y)$ is injective, then f is an immersion at x. If f is an immersion at every point, it is called an immersion.

Example :4.2 The map : $\mathbb{R}^n \to \mathbb{R}^m$ given by

$$(x_1,\dots, x_n) \to (x_1,\dots, x_n, 0,\dots,0),$$

which is assuming n ≤ m, is a immersion. We call this the standard linear immersion.

Lemma :4.3

Let : X → Y be a smooth map between smooth manifolds. If f is an immersion at p ∈ X, then f is an injective immersion in a neighborhood of p.

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Lemma :4.4

Let : X → Y be a smooth map of manifolds. If f is an immersion at any point then dim X$\leq$ dim Y .

This lemma suggests the logic behind the names immersion and submersion. The immersion places a smaller manifold into a larger one while the submersion smoothly packs a larger manifold into a smaller one. In fact ,most of the work of this is related to the Inverse Function Theorem.

Definition :4.5

Two spaces X and Y are locally equivalent if every point of X has a neighborhood which is equivalent to a neighborhood of Y . For instance, the sphere and the plane are locally equivalent.

Lemma :4.6 (Immersion Lemma)

Let : X →Y be a smooth map of manifolds. Suppose f is an immersion at a point p ∈ X. Then f is locally equivalent to the map

$$(x_1,\ldots,x_n) \to (x_1 ,\ldots,x_n , 0, \ldots, 0)$$

at p.

Proof. Since f is a immersion at p, and we assume

$$f : \mathbb{R}^n \to \mathbb{R}^n x \mathbb{R}^l$$

where l = m - n. Then df_p has rank n. Therefore there exist n linearly independent rows of df_p. Without loss of generality we assume the first n rows are linearly independent.

Let $\tilde{f}: \mathbb{R}^n \to \mathbb{R}^n$ by $\tilde{f} = (f_1, \ldots ,f_n)$,and there fore $\det(d\,\tilde{f})_p \neq 0$. Then we have

$$F : \mathbb{R}^n \text{ x } \mathbb{R}^l \to \mathbb{R}^n \text{ x } \mathbb{R}^l$$

by

$$F(x , y) = f(x)+(0, y).$$

Therefore

$$dF_p = \begin{bmatrix} d\tilde{f}_p & 0 \\ * & I \end{bmatrix}$$

and $\det(d\,\tilde{f})_p \neq 0$. By the Inverse Function Theorem, F has a inverse W near p. On a neighborhood of p, we have

$$W . f(x) = W .F(x , 0) = (x , 0).$$

Hence, f is locally equivalent to the map $(x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_n, 0, \ldots, 0)$ at p.

5-SUBMERSIONS

Definition :5.1 If : X →Y carries x to y, and its derivative $df_x : T_x(X) \rightarrow T_y(Y)$ is surjective, f is called a submersion at x. A map that is a submersion at every point is called a submersion.

Example :5.2

The projection p : $\mathbb{R}^n \rightarrow \mathbb{R}^n$ onto the first m coordinates,which is assuming m ≤ n, is a submersion. We call this the standard linear submersion.

Lemma :5.3

Let : X → Y be a smooth map of manifolds. If f is a submersion at any point then dimX ≥dimY .

Example :5.4

The map : $\mathbb{R}$ x (2 , 3) → $\mathbb{R}^2$ given by f(x; y) = y(cos x; sin x) is neither injective nor surjective since f maps the infinite horizontal strip

$\mathbb{R}$ x (2 , 3) onto the open annulus $\{z \in \mathbb{R}^2 : 2 < |z| < 3\}$. Also the Jacobian of f at (x; y) is

$$\begin{bmatrix} -ysinx & cosx \\ ycosx & sinx \end{bmatrix}$$

, which is invertible. Hence f is both an immersion and a submersion.

Lemma :5.5

Let : X -Y be a smooth map between smooth manifolds. If f is a submersion at p ∈ X, then f is a submersion and open map in a neighborhood of p.

Moreover, f is an immersion and a submersion if and only if f is a local diffeomorphism. This is the content of the inverse function theorem.

Lemma :5.6

(Submersion Lemma) Let : X →Y be a smooth map of manifolds. Suppose f is an Submersion at a point p ∈ X. Then f is locally equivalent to the map

$$(x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_m)$$

at p.

Proof. Since f is a Submersion at p, and we assume : $\mathbb{R}^m$ x $\mathbb{R}^l \to \mathbb{R}^m$ where l = n - m. Then d f_p has rank m.

Therefore there exist n linearly independent rows of d f_p. Without loss of generality we assume the first m columns are linearly independent.

Let $\tilde{f}$: $\mathbb{R}^m \to \mathbb{R}^m$ by f(x_1, …. ,x_m)=f(x_1, …. ,x_m, 0, ….,0) so det(d $\tilde{f}$)$_p \neq 0$. Then we have

$$F : \mathbb{R}^m \text{ x } \mathbb{R}^l \to \mathbb{R}^m \text{ x } \mathbb{R}^l$$

by

$$F(x , y) = (f(x,y),y)$$

Therefore

$$dF_p = \begin{bmatrix} d\tilde{f}_p & 0 \\ * & I \end{bmatrix}$$

, and det(d F)$_p \neq 0$. By the Inverse Function Theorem, F is a diffeomorphism near p. Let g be the projection onto $\mathbb{R}^m$ .Then on a neighborhood of p, we have

$$g.F(x , y) = g(f(x,y),y)= f x,y)$$

Hence, f is locally equivalent to the map

$$(x_1 ,\dots,x_n) \to (x_1 ,\dots,x_m)$$

at p.

This lemma is important for Regular Value Theorem.

Theorem :5.7 (Regular Value Theorem) If y ∈ Y is a regular value of

f : X → Y then f^{-1}(y) is a manifold of dimension n-m, since

dim(X) = n and dim(Y) = m.

Proof. Let y be a regular value of f and suppose that x ∈ f^{-1}(y) .By the submersion lemma, there exist local coordinates around x and y such that

$$f(x_1 ,\dots,x_n) = (x_1 ,\dots,x_m)$$

Therefore, in a neighborhood around x, f^{-1}(y)is the set of points (0; _ _ _ ; 0; x m+1; _ _ _ ; xn). Thus, in a small neighborhood of x, we can parameterize f^{-1}(y) by

$$(x_{m+1}, \dots, x_n) \to (0,\dots,0, x_{m+1},\dots,x_n).$$

Hence f^{-1}(y) is a n - m dimensional manifold.

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Our main goal in this chapter is to present Sard's theorem, a statement about the measure of the set of critical values of a smooth map between two manifolds, that the set of critical values of a smooth function f from one manifold to another and it has measure zero. First we must know what it means for a set to have measure zero in a manifold.

6 -EASIER CASES OF SARD'S THEOREM:

Recall: Let : U → V be a smooth map between same dimensional manifolds. We denote that x ∈ U is a regular point if the derivative is nonsingular.For another case, if the derivative is singular, then x is called a critical point.

Recall: A special case of Fubinis theorem. If n = $k + l$, A ⊂ $\mathbb{R}^n = \mathbb{R}^k$ x $\mathbb{R}^l$, A is closed, and all the l-dimensional cross-sections of A have measure zero in $\mathbb{R}^l$, then A has measure zero in $\mathbb{R}^n$.

Note that smooth maps of same dimension take measure zero to measure zero.

Theorem :6.1(Mini-Sard) : Let U ⊂ $\mathbb{R}^n$ be open and let f : U → $\mathbb{R}^m$ be a smooth map. Then if m > n, f(U) has measure zero in $\mathbb{R}^n$.

Proof. We define F : U x → $\mathbb{R}^m$ by F(u , x) = f(u). Map F is smooth since f is smooth. We know that $\mathbb{R}^n$x {0} has measure zero in $\mathbb{R}^m$ because m > n, and therefore its subset U x {0} is also a set of measure zero inU x$\mathbb{R}^{m-n}$. Thus F(U x {0}) has measure zero in $\mathbb{R}^m$.

Since F(U x {0}) =f(U), f(U) has measure zero in $\mathbb{R}^n$.

Theorem :6.2 (Sard's Theorem)

Let : X → Y be a smooth map of manifolds, and let C be the set of critical points of f in X. Then f(C) has measure zero in Y .

Remark The theorem does not claim that the set of critical points in X is a measure zero subset. In fact, if we consider a constant map, then any point in X is a critical point. However, f(X) is still a measure zero set, since any point not in f(X) is a regular value.

We begin with the easiest case in order to prove Sard's Theorem, maps of

$f : \mathbb{R}^1 \to \mathbb{R}^1$.

Theorem :6.3 Let U be an open set in $\mathbb{R}^1$, and f : U → $\mathbb{R}^1$ a continuously differentiable map. Let C be the set of critical points of f such that

$$\boldsymbol{C = \{x \in U : f'(x) = 0\}}$$

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Then f(C) has measure zero in $\mathbb{R}^1$.

Proof. Let I be a closed interval inside U, and we define a function

F : Ix I → $\mathbb{R}^1$ by

$$F(x,y) = \frac{(f(y) - f(x) - f'(x)(y-x)}{|y-x|}$$

if x ≠ y, and F(x; y) = 0

if x = y. Since f is continuously differentiable, the map F is uniformly continuous, and therefore F(x , y) can be made arbitrarily close to zero by making x and y sufficiently close to one another. Given any ∈> 0, we can divide the interval I into n equal subintervals, each of length $\frac{L}{n}$so that

| F(x; y)|<∈.

Then we focus on any one of these subintervals which contains a critical point x of , so thatf'(x) = 0. Then for any other point y in that subinterval,

we have

$$|f(y) - f(x) - f'(x)(y-x)| = |f(y) - f(x)| < \in |y-x| \leq \in \frac{L}{n}$$

Thus,

we have

$$|f(y_1) - f(y_2)| \leq 2 \in \frac{L}{n}$$

hence f contained in an interval with length 2 ∈ $\frac{L}{n}$

There are at most n such subintervals, so the image under f of the critical points which lie in the interval I is contained in a union of intervals of total length 2 ∈ L, for ∈> 0, and therefore has measure zero.

Since the open set U is a countable union of such intervals I, it follows that the image under f of all the critical points also has measure zero. Hence$f$(C) has measure zero in $\mathbb{R}^1$.

Then we have the next case.

Theorem :6.4 Let U be an open set in $\mathbb{R}^2$, and f : U → $\mathbb{R}^1$ is a smooth map. Let C be the set of critical points of f. Then f(C) has measure zero in $\mathbb{R}^1$.

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Proof. Since C is the set of critical points of f , let $C_i \subset C$ be the set of points x ∈ U where all partial derivatives of f of order ≤ i are zero. Then

we have $C \supset C_1 \supset C_2$. We will prove it in three steps:

step 1. The image $f(C - C_1)$ has measure zero.

step 2. The image $f(C_1 - C_2)$ has measure zero.

step 3. The image $f(C_2)$ has measure zero.

proof of step 1.

Since we have $C \supset C_1 \supset C_2$, and therefore C - C1 is singular but non-zero Jacobian. Hence the image f(C - C1) has measure zero, completing step 1.

proof of step 2.

For each point x^* in C_1-C_2, there is $\frac{\partial^2 f}{\partial x_i \partial x_j}$which is not zero at x^*. Therefore the function g(x) = $\frac{\partial f}{\partial x_j}$ vanishes at x^*. but $\frac{\partial g}{\partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$does not. Without lose of generality, we assume i = 1, and we define a map

$$h : U \to \mathbb{R}^2 \text{ by } h(x) = (g(x),\ x_2).$$

The Jacobean matrix dh is nonsingular, so h carries some neighborhood V of x^* diffeomorphically onto an open set V' in $\mathbb{R}^2$.

h carries $C_1 \cap V$ into 0x $\mathbb{R}^1$, and now we consider the function

$$w = f \circ h^{-1} : V' \to \mathbb{R}^1,$$

and let

$$w^* : (0 \times \mathbb{R}^1) \cap V' \to \mathbb{R}^1.$$

Hence, the set of critical values of w* has measure zero in $\mathbb{R}^1$.

But each point in $h(C_1 \cap V)$ is certainly a critical point of w* , since all first order derivatives vanish at such points. Therefore

$$w^* o h(C_1 \cap V) = f(C_1 \cap V)$$

has measure zero. Since C_1- C_2 is covered by countable many such sets V ,it follows that f(C_1- C_2) has measure zero, completing step 2.

proof of step 3. (This is similar to the proof for $\mathbb{R}^1$)

Let I be a closed square inside U with edge length L. We will show that f($C_1 \cap$ U) has measure zero. By Taylors Theorem, the compactness of I, we have that

$$f(x + h) = f(x) + R(x; h)$$

with $|R(x; h)| \leq a|h|^3$

for x $\in C_1 \cap$ Iand x + h $\in$ I, where the constant a depends only on f and I. Then we subdivide I into n^2 sub squares, each of side length $\frac{L}{n}$. Let S be a square of this subdivision which contains a point of C_k.

Then any point of S can be written as x + h, with $|h| \leq \frac{\sqrt{2}L}{n}$,Therefore it follows that f(S) must lie in an interval of length 2a $(\frac{\sqrt{2}L}{n})^3$centered at f(x). There are at most n^2 such sub squares S, hence f($C_2 \cap$ I) is contained in a union of intervals of total length at most $n^2 2a\, (\frac{\sqrt{2}L}{n})^3$ =b/n , for some constant b.

This total length tends to 0 as n → 1, so f($C_2 \cap$I) have measure zero. Since U can be covered with countable many such squares, this shows that f(C_2)has measure zero, completing step 3.

Hence, f(C) has measure zero in $\mathbb{R}^1$.

7- PROOF OF SARD'S THEOREM

Theorem :7.1. (Sard's theorem) Let U be an open set in $\mathbb{R}^n$, and

f : U → $\mathbb{R}^p$ is a smooth map. Let C be the set of critical points of f. Then f(C) has measure zero in $\mathbb{R}^p$.

The theorem is certainly true for n = 0. We will proceed to prove that the theorem is true for n assuming that it is true for n - 1.

Denote$C_1 \coloneqq \{x \in U : (df)_x = 0\}$and for i ≥ 1, we have

$$C_1 \coloneqq \{x \in U : \text{all partial derivatives of f of order } \leq \text{i vanish at x}\}$$

Notice that each C_i is built by taking a finite intersection of sets, and each C_i is closed, so we have a descending sequence of closed sets C ⊃ C_1⊃ C_2…. We then proceed by induction and prove three lemmas.

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we will divide the proof into three steps:

step 1. $f(C_k)$ has measure zero for k ≥ n/p - 1.

step 2. $f(C_k - C_{k+1})$ has measure zero.

step 3. $f(C - C_1)$ has measure zero.

steps 2 and 3 give that $f(C - C_1)$ and $f(C_k - C_{k+1})$ have measure zero. These two steps use the inductive hypothesis and the fact that $\mathbb{R}^n$ is second countable, which means if $\{U_\alpha\}$is set of open sets in $\mathbb{R}^n$, there there is a countable sub collection $\{U_{\alpha k}\}$ so that $\cup_\alpha U_\alpha = \cup_k U_{\alpha k}$. Step 1 uses Taylor'sTheorem to show that if k is sufficiently big, then $f(C_k)$ has measure zero, and step 3 makes use of Fubini's Theorem. These steps combine to give Sard's Theorem by additives of measure zero.

Note that a countable union of sets of measure zero also has measure zero.

Lemma :7.2

For$k \geq \frac{n}{p-1}$, the image $f(C_k)$ has measure zero in $\mathbb{R}^p$.

Proof. Fix such a k. Let S ⊂ U be a cube whose sides are of length δ We will prove that for $k \geq \frac{n}{p-1}$,$f(C_k \cap S)$ has measure zero. Since C_k can be covered by countably many such cubes, this implies $f(C_k)$ has measure zero. From Taylors theorem, the compactness of S and the definition of C_k,

we see that

f(x + h) = f(x) + R(x; h) where

$$R(x,h) = hf'(x) + \frac{h^2}{2} f''(x) + \cdots + \frac{h^n}{n!} f^n(x + \lambda h\,)\ for\ \lambda \in [0,1]$$

$$|R(x,h)| < a|h|^{k+1}\ for\ x \in\ C_k \cap S, x + h\ \in S$$

Here we see a is a constant that depends only on f and S. Now subdivide S into M^n cubes whose sides are of length $\frac{\delta}{M}$. Let S_1 be a cube of the subdivision that contains a point x of C_k. Then any point of S_1 can be written as x + h with

$$|h| < \sqrt{n}\frac{\delta}{M}$$

if $x + h \in S_1$, then It follows that $f(S_1)$ lies in a cube with sides of length $\frac{b}{M^{k+1}}$ centered about f(x), where $b = 2a\left(\sqrt{n}\delta\right)^{k+1}$ is a constant. It follows that $f(C_k \cap S)$ is contained in the union of at most M^n cubes having total volume

$$\leq M^n \left(\frac{b}{M^{k+1}}\right)^p = b^p M^{n-(k+1)p}$$

. Since $k > \frac{n}{p} - 1$ we see total volume $\rightarrow 0$ as $M \rightarrow \infty$.

Thus $f(C_k \cap S)$ is of measure zero, which implies $f(C_k)$ has measure zero, completing step 1.

So we have some l such that $f(C_l)$ has measure zero.

Lemma : 7.3 For $k \geq 1$, the image $f(C_k - C_{k+1})$ has measure zero in $\mathbb{R}^p$.

Proof. This is a similar argument. For each $x \in C_k - C_{k+1}$, one can find some k-th partial derivative of f, denoted by w, that vanishes on C_k but has a first derivative, $\frac{\partial w}{\partial x_1}$, that is not zero at x. Again,

we have a map $h : U \rightarrow \mathbb{R}^n$, and $h(x) = (w(x), x_2, \ldots, x_n)$ maps a neighborhood V of x diffeomorphically onto an open set V'. By construction, h carries $C_k \cap V$ into the hyper plane $\{0\} x \mathbb{R}^{n-1}$.

Again we consider the map $g = f \circ h^{-1}$. Then the critical points of g of type C_k are all in the hyper plane $\{0\} x \mathbb{R}^n$

Let

$$\bar{g}: (\{0\} x \mathbb{R}^{n-1}) \cap V' \rightarrow \mathbb{R}^n$$

be the restriction of g. By induction, the set of critical values of g is of measure zero in $\mathbb{R}^p$. Moreover the critical points of g of type C_k are critical points of $\bar{g}$. It follows that the image of these critical points of g is of measure zero.

Therefore, $f(C_k \cap V)$ is of measure zero. Since $C_k - C_{k+1}$ can be covered by countable many sets V , then $f(C_k - C_{k+1})$ is of measure zero. This shows that $f(C_k - C_{k+1})$ has measure zero, completing step 2.

Thus, $f(C_1)$ has measure zero by $f(C_k - C_{k+1})$ has measure zero.

Example :7.4

If X is a manifold, then the identity mapping : X → X has no critical points, and therefore the set of non regular values has measure zero.

Example :7.5

Let : $\mathbb{R}^2 \to \mathbb{R}$, and $f(x; y) = x^2 + y^2$ is given. We have the derivative zero only at the origin, and $f(0, 0)$ is just a single point, which has measure zero in ℝ.

8-APPLICATIONS OF SARD'S THEOREM

This chapter we will discuss the applications of Sard's Theorem, such as applications to the Whitney Embedding and Immersion Theorems, the existence of Morse functions, and Brouwer's fixed point theorem.

Sard's Theorem: Let f : X → Y be a smooth map of manifolds, and let C be the set of critical points of f in X. Then f(C) has measure zero in Y .

Regular Value Theorem: If y is a regular value of f : X → Y , then the pre image $f^{-1}(y)$ is a sub manifold of X, with dim $f^{-1}(y)$ = dim X - dim Y .

Sard's theorem is important, since combining the Pre image Theorem with Sard's Theorem, we know that since almost all values of a mapping are regular, the pre image of a mapping is almost always a manifold. There are many other useful results which rely on Sard's Theorem, such as the Whitney Embedding Theorem, the Whitney Immersion Theorem, the existence of Morse functions, and the Brouwer's fixed point theorem.

Before solving Whitney's Embedding Theorem , we have to show some corollary , lemma and Whiotney's First Embedding Theorem .

Corollary: 8.1

If $f: N^n \to M^m$ is a smooth math of manifolds of degree n and m respectively, and if n < m the $f(N^n)$ has measure zero in M^m.

Lemma: 8.2

Let M (p , n) denote the set of all real p x n matrices with the product topology $\mathbb{R}^{np}$

and

$$M_k(p, n) \subset M(p, n)$$

denote the subset of matrices with rank k. Then $M_k(p, n)$ has dimension k (p + n - k) as a sub manifold.

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 6 | Issue: 12 |December 2020 || Journal DOI: 10.36713/epra2013 || SJIF Impact Factor: 7.032 ||ISI Value: 1.188

9-WHITNEY'S FIRST EMBEDDING THEOREM

Every n-dimensional differentiable manifold can be embedded in $\mathbb{R}^{2n+1}$as a closed subset.

To prove this theorem, we will first establish some sort of a hierarchy of maps. We will show that under certain conditions any smooth function between two manifolds can be arbitrarily approximated by an immersion, which in turn can be arbitrarily approximated by an injective immersion. The latter, then can be approximated by embedding. We begin by the following theorem,

Theorem: 9.1

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^p$ be a smooth map with 2n ≤ p. Then for any $\in > 0$ there exists a p x n matrix A = (a_{ij})

such that

1. $|a_{ij}| < \in$

2. The map $g: U \subset \mathbb{R}^n \to \mathbb{R}^p$ defined by g (x) = f (x) + A x is an immersion.

Proof. We need to show that there exists a matrix A such that $|a_{ij}| < \in$ and

rank (Dg (x)) = rank (Df (x) + A) = n

Given any $\in > 0$, let Λ= {A_{pxn} : $|a_{ij}| < \in$. Then the measure m(Λ) > 0 .

since m((-∈ , ∈)) = 2∈> 0 and thus

$m(\Lambda) = (2\in)^{pxn} > 0$.

$$g(x) = f(x) + Ax \Rightarrow Dg(x) = Df(x) + A.$$

Consider the subset Ω ⊂ Λsuch that

$$\Omega = \{f \in \Lambda : A = B - Df(x) : \text{rank}(B) < n\}$$

We will show that has measure zero which will then imply the existence of the desired matrix A. To do so, We consider the

$$F_k : M_k(p , n) \times U \to M(p , n) ; F_k(B , x) = B - Df(x)$$

where rank (B) < n. Notice that dim (M_k (p , n) x U) = k (p + n - k) + n.

If we let ∅ (k) = k (p + n - k) + n,

then ∅′ (k) = -2k + p + n > 0 since 2k < 2n ≤ p. So ∅ is increasing. Then

$$k \le n - 1 \Longrightarrow k(p + n - k) \le (n - 1)(p + n - (n - 1))$$

$$= (2n - p) + pn - 1$$

$$< pn$$

$$= \dim(M(p, n))$$

By corollary,

$$m(F_k(M_k(p, n) \times U)) = 0.$$

The existence of the matrix A follows.

We can restate the preceding theorem using the notion of δ −approximations which we define as follows:

Definition: 9.2

Let X be a topological space and (Y , d) be a metric space with metric d. We say that a function g : X → (Y , d) is a δ −approximation of a function

$$f : X \to (Y, d)$$

if for any $\in > 0$, there exists a function $\delta : X \to (0, \in)$ such that

$$d(f(x), g(s)) < \delta(x) \ \forall x \in X$$

Then the previous theorem states that for any n-dimensional manifold M and a smooth function $f: M \to \mathbb{R}^p$ where $2n \le p$, we can find an immersion $g: M \to \mathbb{R}^p$ which is a δ −approximation to f. In fact, more can besaid. If rank (f) = n on some closed subset N ⊂ M, then g can be chosen to be homotopic to f relative to N.

10-WHITNEY EMBEDDING THEOREM

Let M be a smooth manifold of dimension n. A natural question is: which manifolds can be embedded into $\mathbb{R}^m$ as smooth submanifolds? We proceed to our first application.

Theorem :10.1 Any smooth manifold $M_n \subset \mathbb{R}^m$ has an injective immersion into $\mathbb{R}^{2n+1}$.

Proof. Suppose i : M→ $\mathbb{R}^m$ embedding $\pi_a o\ i : M \to V_{a^{\intercal}} \cong \mathbb{R}^{m-1}$. If m = 2n + 1, we are done, so we assume m > 2n + 1. For $a \in \mathbb{R}^m$ such thata $\neq 0$, the projection

$$proj_{a^{\intercal}} v = v - \left(\frac{v.a}{a.a}\right) a.$$

Let π_a be the projection of $\mathbb{R}^m$ to the normal, and it suffices to show that

$$\pi_a|_M : M \rightarrow \mathbb{R}^{m-1}$$

is an injective immersion for at least one a. We will use Sard's Theorem to show that it is true.

Define g : M x M x$\mathbb{R}$ → $\mathbb{R}^m$, and g(x , y , t) = t(x - y), also define

h : TM → $\mathbb{R}^m$ with h(p , v) = $(di)_p(v)$, where (p; v) ∈ TM represents the tangent vector v ∈ $\mathbb{R}^m$ at the point p ∈ M, and

$$di_p : TM_p \rightarrow T\mathbb{R}^m_{i(p)} = \mathbb{R}^m.$$

Ifπ_a : M → $\mathbb{R}^{m-1}$ is not injective, then we have some x , y ∈ M, s ∈ $\mathbb{R}$ such thatx ≠ y and x - y = sa. Therefore,

$$g\left(x, y, \frac{1}{s}\right) = \frac{1}{s}(x - y) = \frac{1}{s}(sa) = a$$

Further more, if π_a is not an immersion, then $d(\pi_a o i)_p v = 0 = \pi_a o di_p(v)$

where π_a is the projection of $\mathbb{R}^m$ and $di_p(v)$ is a vector in $\mathbb{R}^m$, and thus there is some (p , v) ∈ TM such that $di_p(v)$ = s.a for some a. Since M is immersed into $\mathbb{R}^m$, we must have s ≠ 0, so h(v/s) = a. Now it is clear that if a is in neither the range of g nor the range of h, then π_a is the injective immersion.

Since the dimensions of the domains of g and h are 2n+1 and 2n respectively, and m > 2n+1, every point in the range of these functions is a critical value. Thus we can pick almost any a ∈ $\mathbb{R}^m$ and get that $\pi_a : M \rightarrow \mathbb{R}^{m-1}$is an injective immersion.

Note that if M is compact, then an injective immersion is an embedding, so this theorem comes very close to the weak Whitney Embedding Theorem, every manifold M^n can be embedded into $\mathbb{R}^{2n}$.

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ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 6 | Issue: 12 |December 2020 || Journal DOI: 10.36713/epra2013 || SJIF Impact Factor: 7.032 ||ISI Value: 1.188

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