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# MATHEMATICAL MODELING WITH THE SPECTRAL-GRID METHOD OF THE AMPLITUDE OF THE STREAM FUNCTION FOR A PLANE POISEUILLE FLOW

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## ABSTRACT

*To date, extensive experimental material has been accumulated on the behavior of flows in the zone of transition of a laminar flow to a turbulent one, and the available information indicates the great complexity of this phenomenon. The solution to the problem of hydrodynamic stability is of great practical importance, since all hydrodynamic characteristics of a motion essentially depend on whether this motion is laminar or turbulent. The application of numerical methods to the solution of the basic equations - the Navier-Stokes equations - for large Reynolds numbers runs into serious difficulties. They are mainly associated with the presence of a small parameter at the highest derivative and, as a consequence, the appearance in the solution of regions of strong spatial inhomogeneity. Therefore, the requirements for the approximation properties of numerical methods increase dramatically. The stability problem for single-phase hydrodynamic systems is reduced to an eigenvalue problem for the Orr-Sommerfeld equation. The existing methods for modeling the stability problem make it possible to calculate with good accuracy individual eigenvalues of the stability problem and obtain a solution in the regions of inhomogeneity. However, when calculating the spectrum of eigenvalues, as well as own functions, their efficiency turns out to be insufficient. The Orr-Sommerfeld equations contain a small parameter at the highest derivative; therefore, considerable difficulties arise in obtaining approximate solutions close to exact ones.*

**KEYWORDS:** *hydrodynamic stability, Reynolds number, wave number, eigenvalues, own functions, single-phase, Orr-Sommerfeld equations, spectral-grid method, high accuracy, efficiency, efficiency, laminar and turbulent flow*

## INTRODUCTION

Viscosity reflects the property of a liquid to resist the relative movement of adjacent liquid layers. Separate concentric layers slide over one another, and moreover so that the speed everywhere has an axial direction. This kind of movement is called laminar flow (from the Latin word "lamina" - layer) [1-5]. Due to viscosity, liquid particles close to the walls flow more slowly than particles farther from the walls. The flow occurs in an orderly manner in the form of layers moving relative to each other. (layered or laminar flow). However, observations show that at higher Reynolds numbers, the flow ceases to be ordered, i.e. becomes turbulent. The first systematic studies of both, such different forms of flow — laminar and turbulent — were carried out by O. Reynolds [6]. He also carried out an experiment with a colored trickle. Until now, as long as

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the flow remains laminar, the colored liquid introduced into it moves in the pipe in the form of a sharply outlined trickle, but as soon as the flow becomes turbulent, this trickle spreads out and almost uniformly colors the entire liquid moving in the pipe. This shows that in a turbulent flow, transverse motions are superimposed on the main fluid flow, which occurs in the direction of the pipe axis, i.e. movements occurring in a direction perpendicular to the pipe axis. These transverse movements lead to mixing of the moving fluid. As a result of his research, O. Reynolds discovered the similarity law, also named after him. According to the Reynolds similarity law, the transition of a laminar flow to a turbulent one always occurs at approximately the same Reynolds number $Re = \rho UL/\mu$, where $\rho$ is the density of the liquid or gas, μ is the viscosity, U is the characteristic velocity of the main flow, and L is the characteristic length. The Reynolds number at which the transition of a laminar flow to a turbulent one occurs is called the critical Reynolds number $Re_{kp}$. Therefore, those flows for which $Re<Re_{kp}$ are laminar and the same flows for which $Re>Re_{kp}$ are turbulent. Theoretical studies aimed at explaining the above-described phenomenon of the transition of a laminar flow to a turbulent one have been rocking already in the last century. All these studies are based on the idea that laminar flow is subject to some small disturbances. Each theory sought to trace the development in time of perturbations imposed on the main flow, and the form of these perturbations was specially determined in each individual case. The decisive issue to be resolved was to determine whether the disturbances were dying out or growing over time. Damping of disturbances with time should mean that the main flow is stable and vice versa, the growth of disturbances with time should mean that the main flow is unstable and, therefore, a transition to a turbulent flow is possible. In this way, they tried to create a theory of the stability of a laminar flow, which would theoretically calculate the critical Reynolds number $Re_{kp}$ for a given laminar flow. It is now generally accepted that turbulence is a more natural state of fluid flow, and laminar flow occurs only when the Reynolds number is so small that the deviation from this flow tends to attenuate.

## MAIN PART

The study of the hydrodynamic stability problem is reduced to the numerical modeling of the generalized eigenvalue problem for the Orr-Sommerfeld equation [6]:

$$\frac{1}{ik\,\mathrm{Re}}D^2\psi - \left(\left(U(\eta)-\lambda\right)D - \frac{d^2U}{d\eta^2}\right)\psi = 0, \qquad (1)$$

$$\psi_0 < \psi < \psi_1$$

$$\psi(\eta_0) = \frac{d\psi}{d\eta}(\psi_0) = 0, \psi(\eta_1) = \frac{d\psi}{d\eta}(\psi_1) = 0 \qquad (2)$$

with homogeneous boundary conditions (2), which mean impermeability and adhesion requirements. Here $D = \frac{2}{d\psi^2} - k^2$ is the differential operator, $U(\psi)$ is the velocity profile of the main flow, $\psi$ is the coordinate directed across the main flow, $k$ is the wavenumber, $Re$ is the Reynolds number, $\psi(\eta) = \psi_r + i\psi_i$ is the amplitude of the stream function for perturbations, $\lambda = \lambda_r + i\lambda_i$ are the eigenvalues of the problem, where $\lambda_r$ is the phase velocity of the wave disturbance, $\lambda_i$ is the growth coefficient. If $\lambda_i > 0$, then the flow is unstable, if 2 $\lambda_i < 0$, then it is stable. If $\lambda_i = 0$, then the oscillations are neutral stable. From the point of view of the problem of hydrodynamic stability, it is of interest to find the eigenvalues of problem (1) - (2) [5-10]. At the same time, there is another independent problem, the study of the behavior of the own functions of the problem (1) - (2). In this paper, we study the dynamics of changes in the real and imaginary parts of the own functions $\psi(y) = \psi_r + i\psi_i$.

For the numerical simulation of the problem (1) - (2), we use the spectral-grid method (SSM) [15-23]. For this, the interval of integration $[\eta_0,\eta_N]$ is divided into a grid $\eta_0 < \eta_1 < ... < \eta_N$ and thus we obtain $N$ different elements:

$$[\eta_0,\eta_1],[\eta_1,\eta_2],...,[\eta_j,\eta_{j+1}],...,[\eta_{N-1},\eta_N],$$

Differential equation (1) on each of these elements takes the form

$$D^2\psi_j - ikRe[U_j((\eta)-\lambda)D - U_j''(\eta)]\psi = 0, \qquad (3)$$
$$j = 0,1,2,...,N$$

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Boundary conditions (2) are written at the points $\eta_0$ and $\eta_N$:

$$\psi_1(\eta_0)=\frac{d\psi_1}{d\eta}(\eta_0)=0,\psi_N(\eta_N)=\frac{d\psi_N}{d\eta}(\eta_N)=0 \qquad (4)$$

at the points of the partition, we require the continuity of the solution to equation (3) and its derivatives up to the third order. These conditions are of the form

$$\psi_j^{(t)}(\psi_j)=\psi_{j+1}^{(t)}(\psi_j),t=0,1,2,3;\ j=1,2,\ldots,N-1. \qquad (5)$$

where $t$ indicates the order of the derivative. We represent the solutions $\psi_j$ of equation (3) - (4) as a series in the Chebyshev polynomials of the first series. To do this, we map each element $[\eta_j,\eta_{j+1}]$ on the interval $[-1,+1]$ using the following replacement of the independent variable:

$$\eta=\frac{m_j}{2}+\frac{l_j}{2}y,m_j=\eta_j+\eta_{j-1},l_j=\eta_j+\eta_{j-1} \qquad (6)$$

$l_j$ denotes the length of the $j$ th element. After this transformation, equation (3) takes the form

$$D_j^2\psi_j-ik_j\operatorname{Re}_j\left[U_j((y)-\lambda)D_j-U_j(y)\right]\psi_j=0, \qquad (7)$$
$$j=1,2,\ldots,N$$

where

$$D_j=\frac{d^2}{dy_j^2}-k_j^2,k_j=\frac{l_j}{2}k,\operatorname{Re}_j=\frac{l_j}{2}\operatorname{Re}$$

From conditions (4) - (5) we have

$$\psi_j(-1)=0,\frac{d\psi_1}{dy}(-1),l_j^{-t}\psi_j^{(t)}(+1)=l_{j+1}^{-t}\psi_{j+1}^{t}(-1),$$
$$t=0,1,2,3;\ j=1,2,\ldots,N-1, \qquad (8)$$
$$\psi_N(+1)=0,\frac{d\psi_N}{dy}(+1)=0$$

We will seek an approximate solution to problem (7) - (8) at each of the elements in the form

$$\psi_j(y)=\sum_{n=0}^{P_j}a_n^{(j)}T_n(y),$$
$$U_j(y_1^{(j)})=\sum_{n=0}^{P_j}b_n^{(j)}T_n(y_1^{(j)}) \qquad (9)$$
$$y_1^{(j)}=(\cos(\pi l/p_j),l=0,1,2,\ldots,p_j;\ j=1,2,\ldots,N$$

where $T_n(y)$ are Chebyshev polynomials of the first kind, $y_l^{(j)}$ are their nodes, $ap_j$ is the number of polynomials used to approximate the solution on the $j$ - grid element. The expansion coefficients $b_n^{(j)}$ for the function $U_j(y)$ in (9) are determined by the following inverse transformation [7-12]:

$$b_n^{(j)}=\frac{2}{p_jc_n}\sum_{l=0}^{p_j}\frac{1}{c_l}U_j(y_1^{(j)})T_n(y_1^{(j)}),n=0,1,\ldots,p_j,$$

$c_0=c_{pj}=2$, at $m\neq 0,p_j,j=1,2,\ldots,N.$

For the convenience of presenting the SSM, we write equation (7) in operator form, i.e.

$$L_j\psi_j=0,j=1,2,\ldots,N, \qquad (10)$$

where $L_j$ is the differential operator defined by the formula

$$L_j=D_j^2-ik_j\operatorname{Re}_j\left[U_j((y)-\lambda)\right]D_j-U_j''(y)$$

Substituting series (9) into equation (10), we require that the left-hand side of (10) on each of the grid elements be orthogonal to the first $(p_j-4)-m$ Chebyshev polynomials, i.e.

$$(L_j\psi_j, T_n) = 0, n = 0,1,...,p_j-4, j = 1,2,...,N \qquad (11)$$

where $(f,g)=\int_{-1}^{+1} f(x)g(x)(1-x^2)^{-1/2}dx$ is the scalar product on the interval $[-1,+1]$. In addition, we require that the series in Chebyshev polynomials (9) exactly satisfy the boundary conditions and continuity conditions (8). Taking into account the following properties of the Chebyshev polynomials $T_n(\pm1)=(\pm1)^n$ and $T_n'(\pm1)=(\pm1)^{n-2}n^2$, these conditions are written in the form [22-23].

Thus, to determine $\bar{m}=N(p_j+1)$ unknowns $a_n^{(j)}\ (n=0,1,\ldots,p_j,\ j=1,2,\ldots,N)$, we have $\bar{m}$ = $N(p_j+1)$ equations. These equations are: $N(p_j-3)$ - orthogonality equations (11), *4(N–1)* - continuity conditions, and 4 boundary conditions. In the general case, when different numbers of Chebyshev polynomials are given on different elements, we obtain $\bar{m}$ = $(p_1 + p_2 + \ldots + p_N + N)$ equations for determining the same number of unknowns. It is convenient to write the resulting system in matrix form:

$$(A-\lambda B)x = 0 \qquad (12)$$

The complex matrices $A$ and $B$ have a block-diagonal structure, and the vector $x$ contains the coefficients $a_n^{(j)}$ in the expansion (9), i.e.

$$X^T=(a_0^{(1)},a_1^{(1)},\ldots,a_{p1}^{(1)},a_0^{(2)},a_1^{(2)},\ldots,a_{p2}^{(2)},\ldots,a_0^{(N)},a_1^{(N)},\ldots,a_{pN}^{(N)},$$

It is seen that the matrix is degenerate and contains $4N$ zero rows corresponding to the boundary conditions and continuity conditions, since they do not depend on $\lambda$. The corresponding rows of the matrix will contain integers obtained from the values of the Chebyshev polynomials and their derivatives up to the third order at the points −1 or 1. It is impractical to store these integer elements in the complex matrix; moreover, in the complex matrix, the elements corresponding to these rows are equal to zero. Therefore, when compiling a program, complex matrices and are described as follows:

$A(\bar{m}-4N,\bar{m}), B(\bar{m}-4N,\bar{m})$ where $\bar{m}$ is the total number of equations in the algebraic system (12), and $N$ is the number of grid elements in the SSM.

With the help of elementary transformations of columns of matrices and system (12) we will bring to the form [16-23]

$$(AQ-\lambda BQ)(Q-1x)=0, \qquad (13)$$

or

$$(AQ-\lambda BQ)Y=0, \qquad (14)$$

where $Y = Q^{-1}x$,

$$Y^T=(y_0^{(1)},y_1^{(1)},\ldots,y_{p1}^{(1)},y_0^{(2)},y_1^{(2)},\ldots,y_{p2}^{(2)},y_0^{(N)},y_1^{(N)},\ldots,y_{pN}^{(N)},$$

and $Q$ is the corresponding non-degenerate transformation [15-23].

With such a transformation $Q$, the zero rows of the matrix $B$ do not change, and the nonzero rows are transformed according to the transformation $Q$. As a result, a number of equations in system (14) become autonomous:

$$1\cdot y_0^{(1)}=0{,}1\cdot y_1^{(1)}=0{,}4\cdot y_2^{(1)}=0{,}24\cdot y_3^{(1)}=0$$

$$1\cdot y_0^{(2)}=0{,}1\cdot y_1^{(2)}=0{,}4\cdot y_2^{(2)}=0{,}24\cdot y_3^{(2)}=0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$1\cdot y_0^{(N)}=0{,}1\cdot y_1^{(N)}=0{,}4\cdot y_2^{(N)}=0{,}24\cdot y_3^{(N)}=0$$

From this we can see that the first four components of the eigenvector from each grid element are equal to zero,

$$y_0^{(1)}=0,\ y_1^{(1)}=0,\ y_2^{(1)}=0,\ y_3^{(1)}=0,$$
$$y_0^{(2)}=0,\ y_1^{(2)}=0,\ y_2^{(2)}=0,\ y_3^{(2)}=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_0^{(N)}=0,\ y_1^{(N)}=0,\ y_2^{(N)}=0,\ y_3^{(N)}=0,$$

Then the first four rows and the first four columns can be excluded from each block of matrices $AQ$ and $BQ$. The remaining equations give the algebraic system

$$(T-\lambda W)Y=0,$$
$$Y=Q_{\bar{x}}^{-1}, \tag{15}$$
$$\bar{x}=(a_4^{(1)},\ldots,a_{p1}^{(1)},a_4^{(2)},\ldots,a_{p2}^{(2)},\ldots,a_4^{(4)},\ldots,a_{p4}^{(4)},$$

where $W$ is generally a nondegenerate square matrix. Then the order of the matrices $T$ and $W$ will be as follows: $(\overline{m}-4N)*(\overline{m}-4N)$, where $\overline{m}$ is the total number of polynomials in the SSM, i.e.

$$\overline{m}=\sum_{j=1}^{m}(p_j+1)..$$

The $Q$ transformation is used to zero out some elements of the equations obtained from the boundary conditions and continuity conditions. For clarity, these conditions are written in matrices $A$ and $B$. The transformation $Q$ corresponding to the boundary conditions and continuity conditions is formed separately from the matrices $A$ and $B$. The transformation $Q$ is mainly focused on reducing equations with the corresponding boundary conditions and continuity conditions of block-diagonal form. Multiplying (15) on the left by the matrix $W^{(-1)}$, we obtain

$$(D-\lambda E)Y=0, D=TW^{(-1)}. \tag{16}$$

The eigenvalues of system (16) can be found by standard methods. In this work, they were determined using the $QR$-algorithm. Formations $Q$ are the number of rows and columns of complex matrices and are reduced by $4N$, where $N$ is the number of grid elements. At the same time, the high accuracy of the method remains.

## RESULTS AND DISCUSSION

Let the main flow $U(y)$ in (3) be the Poiseuille flow in a flat infinite channel, i.e. $U(y)=1-y^2$. In this case, the characteristic length $L$ is the channel half-width, and the characteristic velocity is the average velocity $U_0$, of the main flow. The Reynolds number is determined by the formula $Re=\rho U_0 L/\mu$, where $\rho$ is the density, $\mu$ is the viscosity of the gas. Boundary conditions (4) for disturbances in the Poiseuille flow have the form

$$\psi(\pm 1)=0,\frac{d\psi}{dy}(\pm 1)=0. \tag{17}$$

Equalities (17) express the usual requirements for impermeability and adhesion. For numerical modeling (3), (17), the above-stated spectral-grid method (SSM) was applied.

The calculation of the spectrum of the Orr-Sommerfeld equation, as well as the calculation of the critical Reynolds number for the Poiseuille flow using the spectral method was carried out in [8], and using the spectral-grid method was carried out in [15, 19, 21]. In these works, using 32 Chebishev polynomials for $k=1, Re=10^4$, the eigenvalue for the unstable mode was found with a high accuracy

$$\lambda = 0.23752649 + 0.00373967i,$$

moreover, the exact knowledge of this mode is

$$\lambda = 0.23752649 + 0.00373967i.$$

The works [15, 19, 21] illustrate the high accuracy and efficiency of the spectral-grid method.

At the same time, in [24], using a difference scheme of the sixth order of accuracy with nodes uniformly spaced relative to the stretched coordinate, the same unstable mode was found with an accuracy

$$\lambda = 0.23752964 + 0.00374248i.$$

with 43 knots as well

$$\lambda = 0.23752650 + 0.00373969i.$$

with 100 mesh points. On a uniform grid, the same scheme gives

$$\lambda = 0.2370744 + 0.00375620i.$$

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with 43 grid points. In [9], it was found

$$\lambda = 0.237413 + 0.003681\,i.$$

using 50 terms of the expansion in symmetric functions.

The results obtained by the spectral-grid method with an accuracy of 7 digits are obtained when using in the approximation such a number of polynomials, which is more than 2 times less than the number of grid nodes in finite-difference methods required to obtain the same accuracy. Taking into account that the matrix methods for finding eigenvalues used both in [9] and [24] require time proportionally to the cube, and memory is the square of the number of polynomials (functions, grid points), the spectral-grid method is much superior to other methods in efficiency.

The eigenvalues and own functions of the Orr-Sommerfeld equation were calculated for various Reynolds numbers lying outside ($\lambda_i < 0$) on ($\lambda_i = 0$) and inside ($\lambda_i > 0$) the neutral curve ($\lambda_i = 0$). The results are shown in tables 1; 2, in table 2 for comparison, some results of work [25] are given.

Table 1

| $Re$ | $K$ | $\lambda_r$ | $\lambda_i$ |
|---|---|---|---|
| 10000 | 1 | 0.2375265 | 0.0037397 |
| 6000 | 1.02071 | 0.2622475 | 0.0003575 |
| 5772.22 | 1.02056 | 0.2640017 | 0.0000000 |
| 5772.12 | 1.02071 | 0.2640200 | -0.0000002 |
| 5250 | 1.02071 | 0.2683958 | -0.0009652 |
| 6000 | 0.848 | 0.2398535 | -0.0050281 |
| 6000 | 0.875 | 0.2436641 | -0.0033966 |

Table 2

| $Re$ | $K$ | Article [25] | |
|---|---|---|---|
| | | $\lambda_r$ | $\lambda_i$ |
| 6000 | 1.02071 | 0.2623 | 0.0003 |
| 5772.12 | 1.02071 | 0.26402 | 0.000 |
| 5250 | 1.02071 | 0.2684 | -0.0010 |

To numerically simulate the amplitude of the stream function for perturbations, one should calculate the vector $Q^{(-1)}x$ for system (13), and thus, the components of the eigenvector

$$X^T = (a_0^{(1)}, \ldots, a_{pN}^{(N)},$$

Then, using these components, sums (9) are calculated and the amplitude of the stream function for disturbances in the Poiseuille flow is determined. The graphs of the stream function amplitude for disturbances of an unstable symmetric mode for the Poiseuille flow at $Re = 6 \cdot 10^3$ and $k = 1.02071$ in Fig. 1 and at $Re = 10^4$ and $k = 1$ in Fig. 2

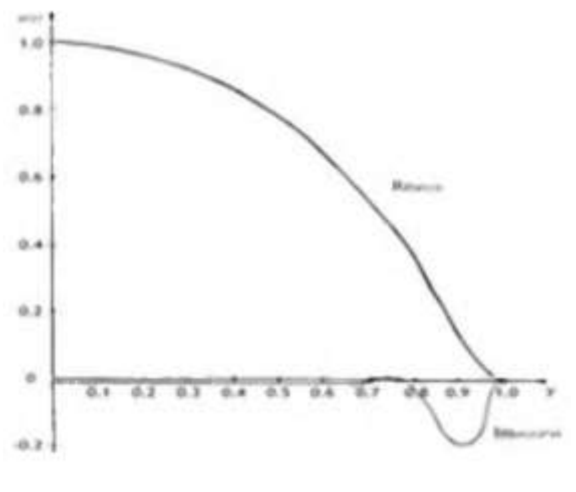

*Fig. 1 Change in the amplitude of the stream function at Re = 6 × 10⁶ and k=1.02071*

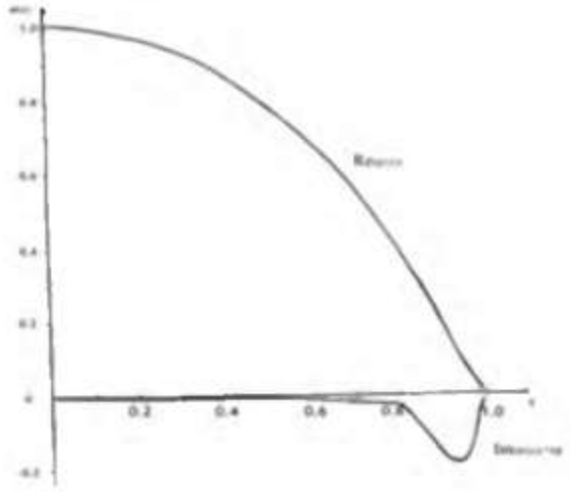

*Fig. 2 Change in the amplitude of the stream function at Re =10⁴ and k=1*

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The calculations performed once again demonstrate the high accuracy and efficiency of the spectral-grid method.

## CONCLUSIONS

1. An algorithm of the spectral-grid method for calculating the amplitude of the stream function has been developed. [6]
2. The eigenvalues and own functions of the plane Poiseuille flow are obtained for various Reynolds numbers and wave numbers. It is shown that the spectral-grid method is very effective in comparison with other methods for solving the problem of hydrodynamic stability.
3. The amplitudes of the stream function for disturbances are investigated and their graphs are plotted.

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