GENERATION OF DIOPHANTINE 3-TUPLES THROUGH MATRIX METHOD

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ABSTRACT

This paper concerns with the formulation of sequences of Diophantine 3-tuples with property \( D(k^2 + 10k - 3) \) through matrix method.

KEY WORDS: Diophantine 3-tuple, Sequence of Diophantine 3-tuples, Matrix application

INTRODUCTION

A theory that can be explained in a regular and systematic way is a pattern. The essence of mathematical calculations is represented by numbers and the theory of numbers can be taught as a set of patterns [1-4]. It is to be noted here that the number pattern is a sequence of numbers establishing a same properties among them. Numbers exhibit fascinating, beautiful and curious varieties of patterns, namely, polygonal numbers, Fibonacci numbers, Lucas numbers, Ramanujan numbers, Kynea numbers, Jacobsthal numbers and so on. Let \( S \) be a set of \( m \) non-zero distinct integers \( (a_1, a_2, a_3, \ldots, a_m) \) have the property \( D(n), n \in Z - \{0\} \) if \( a_ia_j + n \) is a perfect square for all \( 1 \leq i < j \leq m \) or \( 1 \leq j < i \leq m \) and such a set \( S \) is called a Diophantine m-tuple with property \( D(n) \). For illustration see [5-14]. This paper exhibits the construction of sequence of Diophantine 3-tuples with property \( D(k^2 + 10k - 3) \) through matrix method.
METHOD OF ANALYSIS

Initially, construct a diophantine 2-tuple with property \( D(k^2 + 10k - 3) \) and then, extend it to diophantine 3-tuple.

Let \( 4, 7 \) be two distinct integers such that

\[
4 \cdot 7 + k^2 + 10k - 3 = (k + 5)^2,
\]

a perfect square

Therefore, the pair \((4, 7)\) exhibits diophantine double having property \( D(k^2 + 10k - 3) \).

If \( c \) is the 3\textsuperscript{rd} tuple, then it satisfies corresponding double equations

\[
\begin{align*}
4c + k^2 + 10k - 3 &= p^2 \\
7c + k^2 + 10k - 3 &= q^2
\end{align*}
\]

The eliminant of \( c \) in the above two equations leads to

\[
7p^2 - 4q^2 = 3(k^2 + 10k - 3)
\]

Taking

\[
p = X + 4T, \quad q = X + 7T
\]

in (3) and simplifying, we get

\[
X^2 = 28T^2 + k^2 + 10k - 3
\]

which is satisfied by \( T = 1, X = k + 5 \)

From (4) and (1), observe

\[
c = 2k + 21
\]

Note that \((4, 7, 2k + 21)\) is a diophantine triple satisfying the property \( D(k^2 + 10k - 3) \)

The process of obtaining other diophantine triples with property \( D(k^2 + 10k - 3) \) is illustrated below:

Let \( M \) be a \( 3 \times 3 \) square matrix given by

\[
M = \begin{pmatrix}
1 & 0 & 2 \\
0 & 0 & -1 \\
0 & 1 & 2
\end{pmatrix}
\]

Now,
Following the procedure as above, the Diophantine triples obtained are \((7, 4, 2k + 21), (7, 2k + 21, 4k + 52), (19, 4k + 52, 6k + 97), \ldots\ldots\) each with property \(D(k^2 + 10k - 3)\) whose general form \((7, c_{s-1}, c_s)\) is \((7, 7s^2 + (2k - 4)s - 2k + 1, 7(s + 1)^2 + (2k - 4)s - 3)\), \(s = 1, 2, 3, \ldots\ldots\)

It is noted that the triple \((c_{s-1}, c_s + 4, c_{s+1})\), \(s = 1, 2, 3, \ldots\ldots\) forms an arithmetic progression.

**Note 1:**

It is obvious that \((7, 4, 2k + 21)\) is a Diophantine 3-tuples with property \(D(k^2 + 10k - 3)\).

A few numerical illustrations are given in table 1 below:

<table>
<thead>
<tr>
<th>(k)</th>
<th>((4, c_0, c_1))</th>
<th>((4, c_1, c_2))</th>
<th>((4, c_2, c_3))</th>
<th>(D(k^2 + 10k - 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>((4, 7, 21))</td>
<td>((4, 21, 43))</td>
<td>((4, 43, 73))</td>
<td>(D(-3))</td>
</tr>
<tr>
<td>-1</td>
<td>((4, 7, 19))</td>
<td>((4, 19, 39))</td>
<td>((4, 39, 67))</td>
<td>(D(-12))</td>
</tr>
<tr>
<td>-2</td>
<td>((4, 7, 17))</td>
<td>((4, 17, 35))</td>
<td>((4, 35, 61))</td>
<td>(D(-19))</td>
</tr>
<tr>
<td>-3</td>
<td>((4, 7, 15))</td>
<td>((4, 15, 31))</td>
<td>((4, 31, 55))</td>
<td>(D(-24))</td>
</tr>
<tr>
<td>1</td>
<td>((4, 7, 23))</td>
<td>((4, 23, 47))</td>
<td>((4, 47, 79))</td>
<td>(D(8))</td>
</tr>
</tbody>
</table>

It is noted that the triple \((c_{s-1}, c_s + 4, c_{s+1})\), \(s = 1, 2, 3, \ldots\ldots\) forms an arithmetic progression.
Note that \((c_{s-1}, c_s + 7, c_{s+1})\) forms an Arithmetic Progression.

Note 2:

In addition to (4), one may consider the linear transformation given by

\[ p = X - 4T, \quad q = X - 7T \]

For this case, employing the procedure as above one obtains two sets of sequences of Diophantine 3-tuples in which, each triple has the property \(D(k^2 + 10k - 3)\). For simplicity and brevity, the general form of the triple in the sequence of Diophantine 3-tuples is presented:

Set 1: \((4, \alpha_{s-1}, \alpha_s)\) where \(\alpha_{s-1} = 4s^2 + \left(-2k - 18\right)s + 2k + 21, \quad s = 1, 2, 3, \ldots\)

Note that \((\alpha_{s-1}, \alpha_s + 4, \alpha_{s+1})\) forms an Arithmetic Progression.

Set 2: \((7, \alpha_{s-1}, \alpha_s)\) where \(\alpha_{s-1} = 7s^2 + \left(-2k - 24\right)s + 2k + 21, \quad s = 1, 2, 3, \ldots\)

Note that \((\alpha_{s-1}, \alpha_s + 7, \alpha_{s+1})\) forms an Arithmetic Progression.

Remark:

Instead of (5), suppose we have a third order square matrix \(N\) given by

\[
N = \begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & 2 \\
0 & 1 & 2 \\
\end{pmatrix}
\]

Following the procedure presented above, one obtains 4 more sets of Diophantine triples, each with property \(D(k^2 + 10k - 3)\).

To conclude, one may search for other choices of Matrices for the formulation of collections of Diophantine triples with suitable properties.

REFERENCES

1. Dickson L.E. (1952), History of Theory of Numbers, Chelsea publishing company, New York, Vol.II.