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ON THE PAIR OF EQUATIONS $a \pm b = p^3$, $ab = q^2$

M.A.Gopalan
Professor,
Department of Mathematics,
SIGC, Trichy-620002,
Tamilnadu,India.

ABSTRACT

This communication aims at determining pairs of non-zero distinct integers (a,b) such that, in each pair

(i). the sum is a cubic integer and the product is a square integer

(ii). the difference is a cubical integer and the product is a square integer

KEYWORDS: *system of double equations, integer solutions*

1. INTRODUCTION

In the history of number theory, the Diophantine equations occupy a remarkable position as it has an unlimited supply of fascinating and innovating problems [1-9]. This communication concerns with the problem of obtaining two non-zero distinct integers a and b such that

(i). $a+b=p^3$, $ab=q^2$ and

(ii). $a-b=p^3$, $ab=q^2$

2. METHOD OF ANALYSIS

(I) On the system $a+b=p^3$, $ab=q^2$

Let a,b be two non-zero distinct positive integers such that

$$a+b=p^3 \ , \ ab=q^2 \qquad (1,2)$$

where $p,q>0$

The elimination of b between (1) and (2) leads to

$$a^2 - ap^3 + q^2 = 0$$

which is satisfied by

$$a = \frac{1}{2}\left(p^3 + \sqrt{p^6 - 4q^2}\right) \qquad (3)$$

The square root on the RHS is eliminated when

$$q = rs \ , \ p^3 = r^2 + s^2, \quad r > s > 0 \qquad (4,5)$$

and thus, note that

$$a = r^2, b = s^2 \qquad (6)$$

Now, note that the values of r and s should satisfy (5). After some algebra, it is seen that there are two sets of values of r, s given as below:

Set 1: $r = m\left(m^2 + n^2\right), \ s = n\left(m^2 + n^2\right)$

Set 2: $r = m^3 - 3mn^2 \ , \ s = 3m^2n - n^3$

where $m, n \neq 0$

Using Set 1, the values of a, b satisfying (1,2) are given by

$$a = m^2\left(m^2 + n^2\right)^2 \ , \ b = n^2\left(m^2 + n^2\right)^2$$

and in view of set 2, one has

$$a = \left(m^3 - 3mn^2\right)^2 \ , \ b = \left(3m^2n - n^3\right)^2$$

However, it is worth to mention that the square root on the RHS of (3) is also eliminated when

$$q = 2\left(r^2 - s^2\right), \ p^3 = 4\left(r^2 + s^2\right) \qquad (7,8)$$

and we obtain

$$a = 2(r + s)^2, \ b = 2(r - s)^2 \qquad (9)$$

Now, observe that r and s should satisfy (8). It is seen that there are two sets of values to r, s as presented below:

Set 3:

$$r=\alpha^3-3\alpha\beta^2-3\alpha^2\beta+\beta^3$$
$$s=\alpha^3-3\alpha\beta^2+3\alpha^2\beta-\beta^3$$
$$p=\alpha^2+\beta^2$$

Set 4:

$$r=\frac{m\left(m^2+n^2\right)}{2}$$
$$s=\frac{n\left(m^2+n^2\right)}{2}$$
$$p=m^2+n^2$$

where m and n are of the same parity

Employing Set 3 in (9), the values of a and b satisfying (1,2) are given by

$$a=2\left(\alpha^3-3\alpha\beta^2\right)^2\ ,\ b=2\left(\beta^3-3\alpha^2\beta\right)^2$$

and using Set 4 in (9), the corresponding values of a and b satisfying (1,2) are obtained as

$$a=\frac{1}{2}(m+n)^2\left(m^2+n^2\right)^2,$$
$$b=\frac{1}{2}(m-n)^2\left(m^2+n^2\right)^2$$

where in the values of m and n are both even or both odd.

(II) On the system $a-b=p^3\ ,\ ab=q^2$

Let a,b be two non-zero distinct positive integers such that

$$a-b=p^3\ ,\ ab=q^2 \qquad (10,11)$$

Elimination b between (10) and (11), one gets

$$a=\frac{1}{2}\left(p^3+\sqrt{p^6+4q^2}\right) \qquad (12)$$

The square root on the RHS of (12) is eliminated when

$$q=rs\ ,\ p^3=r^2-s^2,\quad r>s>0 \qquad (13,14)$$

and thus,

$$a = r^2, b = s^2 \tag{15}$$

It is to be noted that the values of r and s should satisfy (14). After a few calculations, it is seen that there are two sets of values to r, s as given below:

Set 3: $r = t_{3,p}, \; s = t_{3,p-1}, \; t_{3,p}$ - triangular number of rank p

Set 4:

$$r = 4k^3 + 6k^2 + 3k + 1$$
$$s = 4k^3 + 6k^2 + 3k$$
$$p = 2k + 1$$

Using set 3, the values of a, b satisfying (10,11) are given by

$$a = t_{3,p}^2, \; b = t_{3,p-1}^2$$

and in view of set 4, one has

$$a = \left(4k^3 + 6k^2 + 3k + 1\right)^2$$
$$b = \left(4k^3 + 6k^2 + 3k\right)^2$$

Also, the square- root on the RHS of (12) is eliminated for the following choices of p and q:

Choice (i) $p = 2ks \quad , \quad q = 2s^2\left(k^6 s^2 - 1\right)$

Choice (ii) $p = 2\alpha\beta \quad , \quad q = 2\left(\alpha^6 - \beta^6\right)$

and thus, one obtains

$$a = 2s^2\left(k^3 s + 1\right)^2 \quad , \quad b = 2s^2\left(k^3 s - 1\right)^2$$

and

$a = 2\left(\alpha^3 + \beta^3\right)^2, \; b = 2\left(\alpha^3 - \beta^3\right)^2$ respectively.

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