



CONCEPT OF TRANSVERSALITY AND MORSE THEOREM

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ABSTRACT

In This paper the concept of transversality due to Thom is introduced. This is the smooth counterpart of the notion of general position and is used similarly to extract from messy entanglements their essential geometric content. In this paper it is applied to prove that every function can be approximated by one with a very regular behavior at singularities, at Morse function. It is also used to define intersection Numbers.

The notion of transversality is a smooth equivalent of the notion of general position. For instance, two sub manifolds M^m and V^r of N^n , $n \leq m + r$, are transversal if their intersection looks locally like the intersection in R^n of the subspace of the first m coordinates with the subspace of the last r coordinates.

This geometric idea is properly expressed as transversality of maps and defined in terms of their differentials. This is done in Section 1. The ability of deform maps to a transversal position is one of the most powerful techniques of differential topology. A general theorem in this direction is given here in 2.1

TRANSVERSAL MAPS AND MANIFOLDS

(2.1.1)

Definition: Let $f: M \rightarrow N$, $g: V \rightarrow N$ be two smooth maps. We say that f is transversal to g , $f \pitchfork g$, if when ever $f(p) = g(q)$, then

$$Df(T_pM) + Dg(T_qV) = T_{f(p)}N.$$

Note that this condition is equivalent to the requirement that the composition be surjective.

$$T_pM \xrightarrow{Df} T_{f(p)} \rightarrow \frac{T_{f(p)}N}{Dg(T_qV)}$$

Be surjective.

Obviously, if $\dim M + \dim V < \dim N$, then $f \pitchfork g$ is possible only if $f(M)$ and $g(V)$ are disjoint.



The notation $f \uparrow g$ will be replaced by $f \uparrow V$ whenever V is a submanifold and g an identity map. The meaning of $M \uparrow V$ is also clear.

In certain situations the second map in 2.1.2 is a differential of a map ;hence the composition is also a differential. This is the case when V is a fibre of a smooth fibre bundle N with projection π . Then, if f maps a manifold M into N , the differential of πf is precisely the composition in.

2.1.2. This differential is surjective if and only if the point $\pi(V)$ is a regular value of πf Thus we have:

(2.1.3)

Proposition: *Let $f: M \rightarrow N$, where N is a smooth fiber bundle with projection π , and let F_q be a fiber over a point q . Then $f \uparrow F_q$ if and only if q is regular value of πf .*

Viewing the product $W \times V$ as a bundle over W , we obtain from this and the Brown-Sard Theorem the following:

(2.1.4)

Corollary: *Iff $f: M \rightarrow W \times V$, then there is a dense set of points $q \in V$ such that $f \uparrow W \times (q)$.*

As another corollary we have a characterization of cross sections:

(2.1.5)

Corollary: *Let N be a smooth fiber bundle over M . A submanifold $V \subset N$ is a cross section of the bundle if and only if V intersects every fiber F_q transversely in a single point $s(q)$.*

Proof :The necessity is clear. To prove that the condition is sufficient we have to show that the map $s: M \rightarrow N$ is smooth. To do this, we first note that s is the inverse of $\pi|_V$ and that, by 1.1.3,

$$D(\pi|_V) : T_{s(q)}V \rightarrow T_qM$$

Is surjective. Since $\dim V = \dim M$, $D(\pi|_V)$ is an isomorphism. Now, it follows from the Implicit Function Theorem that the inverse of $\pi|_V$ is smooth.

For example, if V is the image of the imbedding $\mathbb{R} \rightarrow \mathbb{R}^2$ given by

$$t \rightarrow (t^3, t),$$

then V is a smooth submanifold of \mathbb{R}^2 and a continuous section of \mathbb{R}^2 considered as a trivial line bundle over the x axis. But it is not a smooth section: It is not transversal to the y axis.



The notion of transversality generalizes that of a regular value: If $f: M \rightarrow N$ and $q \in N$, then q is a regular value of f if and only if $f^{-1}(q) \neq \emptyset$ and $(f|_{\partial M})^{-1}(q) \neq \emptyset$. Replacing q by a closed submanifold V , we obtain the following generalization of 1.1.7:

(2.1.6)

Proposition: If $f \pitchfork V$ and $(f|_{\partial M}) \pitchfork V$ then $W = f^{-1}(V)$ is a neat submanifold of M . Moreover, $\nu W = f^* \nu V$.

Proof: Let $p \in W$ and $q = f(p)$. By II, 1.2.3(b) there is in N a neighbourhood U of q and a map $h: U \rightarrow \mathbb{R}^r$ such that

$$U \cap V = h^{-1}(0)$$

Moreover, we can identify Dh at q with

$$T_q N \rightarrow T_q N / T_q V.$$

Now,

$f^{-1}(U)$ is an open neighbourhood of p ,

$$f^{-1}(U) \cap W = f^{-1}h^{-1}(0),$$

and both Dhf and $D(hf|_{\partial M})$ are surjective by the assumption. By 1.2.3(b) again, W is a submanifold of M .

Note that $\text{codim}_M(W) = \text{codim}_N(V)$.

Let now d be the dimension of the kernel of the composite map

$$T_W M \xrightarrow{Df} T_p N \xrightarrow{\pi} \nu V = \frac{T_p N}{T_p V}$$

Since $\pi^0 Df$ is surjective, $m - d \geq \text{codim } V$, i.e., $d \leq m - \text{codim } V = \dim W$. On the other hand, $TW \subset \text{Ker}(\pi^0 Df)$; thus $d \geq \dim W$. It follows that $d = \dim W$; hence $\text{Ker}(\pi^0 Df) = TW$. Therefore $f: W \rightarrow V$ induces a bundle map

$$T_W M / TW = \nu W \rightarrow \nu V = T_p N / T_p V$$

A very nice application of 1.1.4 is a simple proof, due to M. Hirsch, of Brouwer's Fixed Point Theorem.

(2.1.7)

Theorem: There is no (continuous) retraction $D^n \rightarrow \partial D^n$.

Proof: Observe first that it is enough to prove that there is no smooth retraction. For if



$$r: D^n \rightarrow \partial D^n .$$

is a continuous retraction, then there is a smooth 1/2-approximation r' to r that is also the identity map on ∂D^n . This is not yet a retraction, but since the origin is not in $r'(D^n)$ we can compose r' with the projection from the origin to obtain a smooth retraction.

Suppose now that $r: D^n \rightarrow \partial D^n$ is a smooth retraction, let $p \in \partial D^n$ be a regular value of r , and let L be the connected component of $r^{-1}(p)$ containing p . Since $r^{-1}(p)$ is a neat submanifold, L is an arc with end points p and $q, p \neq q$ and $q \in \partial D^n$. This implies $p = r(q) = q$, a contradiction.

The notion of transversality already appeared, in disguise, in the definition of neat submanifolds: 1.2.8.1 means nothing else but that $M \pitchfork \partial N$. Moreover, as we have seen, this condition characterizes neat submanifolds.

The following theorem, which for simplicity is stated for closed manifolds only, provides the expected geometric justification of the definition of transversality.

(2.1.8)

Theorem : Let M^m and V^r be closed transversal submanifolds of N^n and let $p \in M \cap V$. If $n \leq m + r$, then there is in N a chart U about p in which $U \cap M$ is represented by the space of the first m coordinates and $U \cap V$ is represented by the space of the last r coordinates.

Proof : We will prove this in the special case $\dim N = m + r$. we can say that there is a chart U in N about p such that $U \cap M$ corresponds to the space of the first m coordinates. We will simply identify this chart with $\mathbb{R}^m \times \mathbb{R}^r$. The part of V lying in it can then be represented by an image of \mathbb{R}^r under an imbedding

$$f: \mathbb{R}^r \rightarrow \mathbb{R}^m \times \mathbb{R}^r,$$

where

$$f(y) = (\alpha(y)\beta(y)) \text{ and } f(0) = 0 = p.$$

The transversality assumption means that the Jacobean of β is of rank r at 0. Now, consider the map

$$g: \mathbb{R}^m \times \mathbb{R}^r \rightarrow \mathbb{R}^m \times \mathbb{R}^r$$

given by

$$g(x, y) = (x + \alpha(y)\beta(y)) \text{ } x \in \mathbb{R}^m, Y \in \mathbb{R}^r.$$



Note that g at 0 is of rank $m + r$; hence it is a chart if restricted to a suitably small neighbourhood U of 0 in $\mathbb{R}^m \times \mathbb{R}^r$.

Since

$$g(0, y) = f(y), \quad g(x, 0) = (x, 0),$$

it is precisely the chart we were looking for.

(2.1.9)

Corollary: Let M^m, V_1^r, V_2^r be submanifolds of $N^n, n = m + r$. Suppose that V_1, V_2 intersect M in the same point p and that this intersection is transversal. Then there is an isotopy of N that keeps M fixed and brings V_1 to coincide with V_2 , in a neighbourhood of p .

Proof :By 1.1.6 there is a chart $U = \mathbb{R}^m \times \mathbb{R}^r$ in N about p that intersects M in $\mathbb{R}^m \times 0$ and V_1 in $0 \times \mathbb{R}^r$. A sufficiently small chart $U_2 = \mathbb{R}^r$ about p in V_2 is represented in U as an imbedded \mathbb{R}^r transversal to $\mathbb{R}^m \times 0$ and intersecting it in the origin.

Now, choosing “straighten” U_2 by an isotopy so that it becomes a linear subspace of $\mathbb{R}^m \times \mathbb{R}^r$ still transversal to $\mathbb{R}^m \times 0$. An obvious isotopy brings it then to coincide with $0 \times \mathbb{R}^r$. These isotopies restricted to the unit disc D^r in U_2 and set to be stationary on M extend to an isotopy of N that sends

$$D^r \subset V_2 \text{ to } V_1.$$

2.2

TRANSVERSALITY THEOREM

The concept of transversality derives its strength from the theorem of Thom asserting that if $f : M \rightarrow N$ and V is a submanifold of N , then f can be approximated by maps transversal on V . We will obtain the theorem of Thom as a consequence of the following fundamental theorem:

(2.2.1)

Theorem: Let ξ be a vector bundle over V and let $f : M \rightarrow N = E(\xi)$ be a smooth map. Then there is a section $s : V \rightarrow E$ such that $f \uparrow s$.

Before proving 2.2.1 we will consider the following situation: We are given a fiber bundle ζ with projection π and base E , and maps

$$f: M \rightarrow E, g: V \rightarrow E(\zeta).$$

This yields a diagram

$$\begin{array}{ccc}
 M_1 = E(f * \zeta) & \xrightarrow{f_1} & E(\zeta) \xleftarrow{g_1} V \\
 \downarrow & & \downarrow \swarrow \\
 M & \xrightarrow{f} & E
 \end{array}$$

Dig : 2.2 (a)

where M_1, π_1 are, respectively, the total space and the projection of the induced bundle $f * \zeta$, f_1 is the natural map, and $g = \pi g_1$. We have:

(2.2.2)

Proposition: If $f_1 \uparrow g_1$ then $f \uparrow g$.

Proof : Suppose that $f(p) = g(q)$. We have to show that

$$Df(T_p M) + Dg(T_q V) = T_{f(p)} E.$$

Note first that there is a point P_1 in M_1 such that $f_1(p_1) = g_1(q)$ and

$\pi_1(p_1) = p$. The assumption $f_1 \uparrow g_1$, means that

$$Df_1(T_{p_1} M_1) + Dg_1(T_q V) = T_{f_1(p_1)} E(\zeta)$$

Now apply $D\pi$ to both sides of this and note that $D\pi, D\pi_1$ are both surjective. Thus, by commutativity,

$$\begin{aligned}
 T_{f(p)} E &= D\pi(T_{f_1(p_1)} E(\zeta)) \\
 &= D\pi f_1(T_{p_1} M_1) + D\pi g_1(T_q V) \\
 &= Df(T_{\pi_1(p_1)} M) + Dg(T_q V)
 \end{aligned}$$



Proof of 2.2.1 Assume first that ξ is trivial, i.e., $E = V \times R^k$. Then 1.3.1 yields a (dense set of points) q in R^k such that $f \uparrow V \times \{q\}$. Of course, each such $V \times \{q\}$ can be interpreted as a section of ξ , which proves 2.2.1 for a trivial bundle ξ .

In the general case there is a bundle η such that $\zeta = \xi \oplus \eta$ is trivial. There is a natural projection π of $E(\zeta)$ onto E , which is a projection of a vector bundle. Thus we have the left part of diagram 2.2(a)

Since ζ is trivial, there is a section g_1 transverse to f_1 . This completes the diagram 2.2(a) and we can apply 2.2.2 to deduce that $g \uparrow f$: It remains to be shown that g is a section of ξ . This follows from the fact that g_1 is a section and the obvious commutativity of the diagram

$$\begin{array}{c}
 E(\zeta) \rightarrow V, \\
 \downarrow \quad \nearrow \\
 E
 \end{array}$$

where all maps are projections of vector bundles.

2.3

THE MORSE LEMMA

Let X be a smooth manifold, $f: X \rightarrow \mathbb{R}$ a smooth function. Since \mathbb{R} is one dimensional as a manifold, the derivative of f must have rank zero or one at each $p \in X$.

Thus a critical point p of f is simply a point for which all the partial derivatives of f vanish. Relative to any coordinate system we have:

$$\left(\frac{\partial f}{\partial x_1} \right)_p = \dots = \left(\frac{\partial f}{\partial x_n} \right)_p = 0$$

However, not all critical points are created equal. The following tool encodes the critical information that we will use to construct normal forms for the structure of functions near most critical points.

Definition 2.3.1.

Let $f: X \rightarrow \mathbb{R}$ be a smooth function.



(1) The *Hessian* of f at p , with respect to local coordinates x_1, \dots, x_n , is the matrix

$$H_p f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_p$$

of second order partial derivatives:

$$\begin{bmatrix} \left(\frac{\partial^2 f}{\partial x_1^2} \right)_p & \cdots & \left(\frac{\partial^2 f}{\partial x_n \partial x_1} \right)_p \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial^2 f}{\partial x_n \partial x_1} \right)_p & \cdots & \left(\frac{\partial^2 f}{\partial x_n^2} \right)_p \end{bmatrix}$$

(2) A critical point p of f is degenerate if $\det H_p(f) = 0$. Otherwise, p is non-degenerate.

(3) The index of f at a non-degenerate critical point p is the maximum dimension of a vector subspace of \mathbb{R}^n on which $H_p(f)$ is negative definite.

Remark 2.3.2 $H_p(f)$ is negative definite on V if the corresponding bilinear form

$$H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

is negative definite, i.e. $H(v, v) < 0$ for every non-zero $v \in V$.

Equivalently, the index can be viewed as the number of negative Eigen values of the non-singular Hessian matrix.

Note that we have defined the Hessian of f at p in a way that depends on the

particular chart chosen at p . There also exists an invariant formulation of the Hessian using the concept of intrinsic derivative. While we have avoided the latter approach for simplicity, we must now do a little work to verify that the degeneracy and index of a function at a point are well-defined notions.

Proposition 2.3.4. *The degeneracy and index of f at p do not depend on the coordinates chosen on X .*

Proof. Let $A = H_p(f)$ be the Hessian matrix with respect to the coordinates

x_1, \dots, x_n given by a chart (U, φ) of X at p .

Let

$$\varphi: \varphi(U) \rightarrow \varphi(U)$$

be a change of coordinates defined by



$$\varphi(x_1, \dots, x_n) = (y_1(x_1, \dots, x_n), \dots, y_n(x_1, \dots, x_n)).$$

Then the matrix $P = (d\varphi)_0$ is non-singular, and the matrix of the Hessian of f at p with respect to the coordinates y_1, \dots, y_n is given by $(P^{-1})^T A P^{-1}$.

The latter claim is an exercise in quadratic forms, namely that a change of coordinates replaces a quadratic form with matrix A by a quadratic form with matrix $B^T A B$, where B is non-singular. Clearly A is singular iff $B A B^T$ is singular. And if A is non-singular, then A and $B A B^T$ have the same index by Sylvester's Law .

By the Submersion Lemma, a smooth function is locally equivalent at a regular point to projection onto the first coordinate. The Morse Lemma provides normal forms for the local behavior of smooth functions at non-degenerate critical points.

Theorem 2.3.5 (Morse Lemma). *Let $f: X \rightarrow \mathbb{R}$ be a smooth function, $p \in X$ a non-degenerate critical point of f , and λ the index of f at p . Then near p , f is equivalent to the map*

$$(x_1, \dots, x_n) \rightarrow x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2$$

Our proof of the Morse Lemma fleshes out the sketch given by Milnor and will require the following calculus result.

Lemma 2.3.6. *Let f be a smooth function on some convex region $V \subset \mathbb{R}^n$, with $f(\mathbf{0}) = 0$. Then there exist smooth functions (g_1, \dots, g_n) on V with*

$$f(x_1, \dots, x_n) = \sum_{i=1}^n x_i g_i(g_1, \dots, g_n)$$

and

$$g_i(\mathbf{0}) = \frac{\partial f}{\partial x_i}(\mathbf{0}) \text{ for every } 1 \leq i \leq n$$

Proof: Let

$$\begin{aligned} & f(x_1, \dots, x_n) \\ &= \int_0^1 \frac{df}{dt}(x_1 t, \dots, x_n t) dt \end{aligned}$$



$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_1 t, \dots, x_n t) dt$$

by the fundamental theorem of calculus and the chain rule. Note that convexity guarantees that the above integral is defined. So it suffices to set

$$g_1(x_1, \dots, x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(x_1 t, \dots, x_n t) dt$$

where $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ again follows from the fundamental theorem of calculus.

Proof of the Morse Lemma. In *Part A*, we will prove the existence of a change of coordinates on the domain which yields the diagonalized quadratic form

$$f(p) \pm x_1^2 \pm \dots \pm x_n^2.$$

In *Part B* we will show that the index of

$$f(p) - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2 \text{ at } 0 \text{ is } \lambda.$$

Part A. We can assume without loss of generality that $0 = p = f(p)$ and $X = \mathbb{R}^n$, since we are only concerned with local equivalence., there exist smooth functions (g_1, \dots, g_n) on \mathbb{R}^n with

$$f(x_1, \dots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \dots, x_n)$$

and

$$g_i(0) = \frac{\partial f}{\partial x_i}(0)$$

Since $0 \in \mathbb{R}^n$, is a critical point, we have $\frac{\partial f}{\partial x_i}(0) = 0$

for every $1 \leq i \leq n$. Therefore, this time to each of the g_i . So there exist smooth functions h_{ij} , $1 \leq i, j \leq n$, such that

$$g_i(x_1, \dots, x_n) = \sum_{j=1}^n x_j h_{ij}(x_1, \dots, x_n)$$

Substitution gives

$$f(x_1, \dots, x_n) = \sum_{i,j=1}^n x_i x_j h_{ij}(x_1, \dots, x_n)$$

Furthermore, we can assume $h_{ij} = h_{ji}$ (otherwise replace each h_{ij} with

$$1/2 (h_{ij} + h_{ji})).$$



Differentiating gives $\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = 2h_{ij}(0)$, so the matrix

$$(h_{ij}(0)) = \left(\frac{1}{2} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right) \right).$$

By hypothesis, 0 is a non-degenerate critical point of f , so we conclude that $(h_{ij}(0))$ is non-singular.

We now proceed as in the proof of the diagonalization of quadratic forms.

Suppose by induction that there exist coordinates u_1, \dots, u_n on a neighbourhood U_1 of 0 such that

$$f(u_1, \dots, u_n) = \pm u_1^2 \pm \dots \pm u_{r-1}^2 + \sum_{i,j=r}^n u_i u_j H_{i,j}(u_1, \dots, u_n)$$

on U_1 , where the H_{ij} are smooth functions with $H_{ij} = H_{ji}$ and the matrix $(H_{ij}(0))$ non-singular. We have already established the base case $r = 0$.

For the induction step, we first show that we can make $H_{rr}(0) \neq 0$ by a non-

singular linear transformation on the last $n - r + 1$ coordinates. The proof works the same for any r , so for simplicity

let $r = 1$. If we have $H_{ii}(0) \neq 0$ for some $1 \leq i \leq n$ then we are done by transposing u_1 and u_i . Otherwise, since $(H_{ij}(0))$ is non-singular,

there exists some $H_{ii}(0) \neq 0$ with $i \neq j$. Through a pair of transpositions, we can assume $H_{11}(0) = 0$ and $H_{12}(0) =$

$H_{21}(0) \neq 0$. We define a new set of coordinates u'_1, \dots, u'_n on U_1 by

$$u'_1 = \frac{1}{2}(u_1 + u_2)$$

$$u'_2 = \frac{1}{2}(u_1 - u_2)$$

$$u'_i = u_i \text{ for } i > 2$$

This linear transformation is invertible with inverse given by

$$u_1 = (u'_1 - u'_2)$$

$$u_2 = (u'_1 + u'_2)$$

$$u_i = u'_i \text{ for } i > 2$$

Substituting in these new coordinates and regrouping terms, we have



$$f(0) = \sum_{i,j=1}^n u_i' u_j' H'_{ij}(0)$$

with

$$H'_{11}(0) = H_{12}(0) + H_{21}(0) = 2H_{12} \neq 0$$

So without loss of generality we assume $H_{rr}(0) > 0$ (sending u_r to $-u_r$ if necessary). Then there exists a neighbourhood

$U_2 \subset U_1$ on H_{rr} is positive. We define a new set of coordinates v_1, \dots, v_n by

$v_i = u_i$ for $i \neq r$.

$$v_r = \sqrt{H_{11}(u_1, \dots, u_n)} \left[u_r + \sum_{i>r} \frac{u_i H_{ir}(u_1, \dots, u_n)}{H_{rr}(u_1, \dots, u_n)} \right]$$

Note v_r is well-defined and smooth on U_2 . A simple calculation shows

$$\frac{\partial v_r}{\partial u_r} = \sqrt{H_{rr}}$$

So $\frac{\partial v_r}{\partial u_r}(0) \neq 0$ It follows from the Inverse Function Theorem that the change of coordinates map ϕ defined by

$$(u_1, \dots, u_n) \rightarrow (v_1(u_1, \dots, u_n), \dots, v_n(u_1, \dots, u_n))$$

is a diffeomorphism in some sufficiently small neighbourhood $U_3 \subset U_2$ of 0. Then

$$\begin{aligned} f &= \pm u_1^2 \pm \dots \pm u_{r-1}^2 + \sum_{i,j=r}^n u_i u_j H_{ij} \\ &= \pm u_1^2 \pm \dots \pm u_{r-1}^2 + \left[u_r^2 H_{rr} + 2u_r \sum_{i>r} u_i H_{ri} + \sum_{i>r} \frac{u_i^2 H_{ii}}{H_{rr}} \right] \\ &\quad + \sum_{i>r} \frac{u_i^2 (H_{ii} - H_{ii})}{H_{rr}} + \sum_{i,j>r, i \neq j} u_i u_j H_{ij} \end{aligned}$$

The term in brackets is v_r^2 so it is clear that we can choose smooth functions $H'_{ij}(v_1, \dots, v_n)$ for $i > r$ so that

$$f(v_1, \dots, v_n) = \sum_{i=1}^r \pm v_i^2 + \sum_{i,j>r} v_i v_j H'_{ij}(v_1, \dots, v_n)$$

with $H'_{ij} = H'_{ji}$



A fundamental idea due to M. Morse was to single out a class of functions with a particularly nice behavior at critical points and to show that they form a dense set, “Nice behavior” means that at critical points they behave like- i.e., are equivalent to-one of the quadratic functions $\sum \delta_i x_i^2$ at 0, $\delta_i = \pm 1$ In particular, the list of possibilities is- up to equivalence-finite.

As usual, we prefer an invariant definition and the easiest way is to work in the cotangent space. Recall that, given

$$f: M \rightarrow \mathbb{R}, df: M \rightarrow T^*M$$

is the section of the cotangent bundle given at

$$p \in M \text{ by } df(X) = X(f), X \in T_pM.$$

(2.4.1) Definition: We say that $p \in M$ is critical if $df = 0$ at p , i.e., if df intersects the zero section M_0 of the cotangent bundle above p . We say that p is a non-degenerate critical point if this intersection is transversal. A function f which has only non degenerate critical points, that is, such that $df \uparrow M_0$ is called a Morse function.

(2.4.2)

Lemma: Critical points of a Morse function are isolated.

We will delay for a moment the investigation of the local behavior of Morse functions and begin by showing that there are, indeed, a lot of them.

(2.4.3)

Lemma :Let M be a submanifold of \mathbb{R}^k and let $f: M \rightarrow \mathbb{R}$. There is a dense set of linear functions $L: \mathbb{R}^k \rightarrow \mathbb{R}$ such that $f - L$ restricted to M is a Morse function.

Proof We will build a diagram of spaces and maps in the following way:

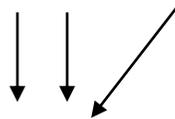
Begin with the cotangent bundle of \mathbb{R}^k restricted to M , i.e., $T^*\mathbb{R}^k/M$. This is also a bundle over T^*M with the projection π .

Then the map

$$df: M \rightarrow T^*M$$

yields the induced bundle with total space E and all this forms the

$$E \xrightarrow{g} T^*R^k|_M \xleftarrow{dL|_M} M$$



$$M \xrightarrow{df} T^*M$$

To get the triangle on the right, note that $T^*R^k|_M$ is a trivial bundle, hence by 1.3.1 there is a dense set of constant sections $M \times \{q\}$ that are transverse to g . A constant section is a differential of a linear map $L: R^k \rightarrow R$. Thus to complete the diagram we choose as L a linear map such that $dL|_M \uparrow g$ and observe that

$$\pi^0 dL|_M = d(L|_M).$$

Now, 2.2.2 implies that $df \uparrow d(L|_M)$, i.e., that $d(f - L|_M)$ is transversal to the zero section.

(2.4.4)

Theorem : *Given $f: M \rightarrow R$ and $\epsilon > 0$, there is a Morse function $g: M \rightarrow R$ such that $|f - g| < \epsilon$.*

Proof: Consider M as a submanifold of the unit ball in an R^k and take as L a linear function such that $|L| < \epsilon$ in the ball.

Now, let M be a manifold with compact boundary and suppose that $\partial M = V_0 \cup V_j$, where the V_j are disjoint and compact.

(2.4.5)

Theorem: *There is a Morse function $f: M \rightarrow I$ such that:*

(a) *f has no critical points in a neighbourhood of ∂M ;*

(b) *$f^{-1}(i) = V_j, j = 0, 1$.*

Proof: Let $\partial M \times [0, 1) \subset M$ be a collar of ∂M . there is a smooth function $g: M \rightarrow I$ with the following properties:



$$g(x, t) = t \quad \text{for } (x, t) \in V_0 \times [0, 1/2],$$

$$g(x, t) = 1 - t \quad \text{for } (x, t) \in V_1 \times [0, 1/2],$$

$$1/4 < g(x) < 3/4 \text{ elsewhere.}$$

Then g has properties (a) and (b) but is not necessarily Morse. To obtain a Morse function we assume that M is a submanifold of the unit ball in an \mathbb{R}^k and consider the function $f = g + \mu L$, where $\mu: M \rightarrow \mathbb{R}$ is smooth, equals 0 in $\partial M \times [0, 1/2]$ and equals 1 in $M - \partial M \times [0, 1/2]$ and L is a still to be chosen linear map of \mathbb{R}^k .

Clearly, f satisfies (a) and, if $|L| < 1/4$ in M , then it satisfies (b) as well.

Assume that some Riemannian metric is given in T^*M .

Since

$$|d(\mu L)| \leq |d\mu||L| + \mu|dL|$$

we see that by taking L “small” we can make $|d(\mu L)|$ (as small as we want in the compact set $\partial M \times [0, 1/2]$). In particular, since $|dg|$ is bounded away from 0 in this set, we can achieve that

$$|d(g + \mu L)| \geq |dg| - |d(\mu L)| > 0$$

in $\partial M \times [0, 1/2]$,

i.e., that f has no critical points there. Then, if L is such that $g + L$ is Morse in M , the same is true of $f = g + \mu L$.

2.5 Morse Functions are Generic

We have seen that Morse functions have simple local behavior, but this result would be of little use if few functions satisfied the Morse property. In this section, we will see that in fact almost all functions are Morse. First, we had better make almost all precise.

Definition 2.5.1 A property P is *generic* if the set

$$\{f \in C^\infty(X, Y) \mid f \text{ has property } P\}$$

is a residual subset of $C^\infty(X, Y)$.



The goal of this section is to show that the quality of being Morse is a generic property of smooth functions. Our strategy will be to translate non-degeneracy into a transversality condition on jets and apply the Thom Transversality Theorem.

S_r is the smooth submanifold of $J^1(X, \mathbb{R})$ consisting of those jets which drop rank by r . For a smooth function $f: X \rightarrow \mathbb{R}$, only S_0 and S_1 can be non-empty.

Moreover,

$$j^1 f: X \rightarrow J^1(X, \mathbb{R})$$

maps critical points to S_1 and regular points to S_0 . The following proposition provides the key link between non-degeneracy and transversality.

Definition 2.5.2 Let $f: X \rightarrow Y$ be a smooth map of manifolds. f is *stable* if the equivalence class of f is open in $C^\infty(X, Y)$, with the C^∞ topology.

Informally, f is stable if all nearby maps look like f . Note that if f is stable, then all of its differentially invariant properties are unchanged by sufficiently small perturbations of f .

2.6 NEIGHBORHOOD OF A CRITICAL POINT

There remains to investigate the behavior of a Morse function in a neighbourhood of a critical point.

Suppose that p is a critical point of $f: M \rightarrow \mathbb{R}$ and choose a local chart at p . The Hessian of f at p is the matrix of second derivatives of f at p . It depends on the choice of the local chart. However:

(2.6.1)

Lemma: *Let p be a critical point of f . Then p is non-degenerate if and only if the Hessian of f at p is of maximal rank.*

Proof: A choice of a chart in a neighbourhood U of p also gives a trivialization of the cotangent bundle restricted to U , that is, a projection

$$\phi: T^*M|_U \rightarrow T_p^*M.$$



p is non-degenerate if and only if $0 \in T_p^*M$ is a regular value of $\emptyset df$, i.e., if the differential of this map at p is surjective.

In the chosen local coordinate system this means that the Jacobean of $\emptyset df$ is to be of maximal rank. However, the map $\emptyset df$ simply assigns to every point the coordinates of df at this point; thus its Jacobean is the Hessian of f at P .

(2.6.2)

Proposition: *Suppose that p is a non-degenerate critical point of f :*

Then in some system of local coordinates at p , f is given by

$$f(p) + \sum_i \delta_i x_i^2, \delta_i = \pm 1$$

Proof: Let f be a real valued function defined in a neighbourhood of $0 \in \mathbb{R}^m$.

Suppose that the Hessian of f at 0 is of maximal rank and that $f(0) = 0$.

We have to show that there is a diffeomorphism h of a neighbourhood of 0

such that

$$fh(x_1, x_2, \dots, x_m) = \sum_{i \leq k} x_i^2 - \sum_{i > k} x_i^2$$

This will be done in two steps. In the first we show that

$$f(x) = \sum_{i,j} h_{ij} x_i x_j \dots \dots (i)$$

where the h_{ij} are some functions of x and $h_{ij} = h_{ji}$. Thus f looks like a symmetric bilinear form-but with variable coefficients-which suggests that we should try to adapt one of usual procedures of diagonalization of such forms to our situation. This works, and that is the second step of the proof. Now the details.

Since f has a critical point at 0 we have, by A ,2.2,

$$f(x) = \sum_i h_i(x) x_i$$

where $h_i(0) = \left(\frac{\partial f}{\partial x_i}\right)(0) = 0$.

We can apply the same lemma once more to h_i to get

$$h_i = \sum_j h_{ij} x_j.$$



Now, setting

$$h_{ij} = \frac{1}{2}(h_{ij} + h_{ji}) = (h_{ij} + h_{ji})$$

we finally obtain equation (i).

The diagonalization of f is now done inductively. Suppose that in some chart f is already in the form

$$f(x) = \pm x_1^2 + \dots \pm x_{k-1}^2 + \sum_{i,j \geq k} h_{ij} x_i x_j \quad h_{ij} = h_{ji}$$

Through a linear change of coordinates we can achieve that $h_{kk}(0) \neq 0$

hence $h_{kk}(x) \neq 0$ in a certain neighbourhood U of 0. Consider the transformation $F: U \rightarrow \mathbb{R}^m$ given by

$$y_i = x_i \text{ for } i \neq k,$$

The Jacobean of F at 0 does not vanish: Its determinant equals $|h_{kk}(0)|^{1/2}$.

Therefore F is a diffeomorphism in a neighbourhood $V \subset U$ of 0 in \mathbb{R}^m .

Since

$$f \circ F^{-1}(y) = \sum_{i \leq k} \pm y_i^2 - \sum_{i,j > k} \frac{h_{i,j} h_{j,k}}{h_{kk}} y_i y_j$$

this concludes the inductive step.

The number of minus signs in this local representation off at a critical non-degenerate point p does not depend on the choice of chart; it is called the index of p .

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