FASCINATING DIOPHANTINE 3-TUPLES FROM THE PAIR OF INTEGERS \( \{u, v\} \)

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ABSTRACT
This paper concerns with the construction of sequences of diophantine 3-tuples \((a, b, c)\) from the pair of integers \(\{u, v\}\) such that the product of any two elements of the set added by \(D(\alpha)^2k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2\) is a perfect square.

KEYWORDS: Diophantine 3-tuples, sequences of triples

INTRODUCTION
The problem of constructing the sets with property that product of any two of its distinct elements is one less than a square has a very long history and such sets have been studied by Diophantus. A set of \(m\) distinct positive integers \(\{a_1, a_2, a_3, \ldots, a_m\}\) is said to have the property \(D(n), n \in \mathbb{Z} - \{0\}\) if \(a_i a_j + n\) is a perfect square for all \(1 \leq i < j \leq m\) or \(1 \leq j < i \leq m\) and such a set is called a Diophantine \(m\)-tuple with property \(D(n)\).

Many Mathematicians considered the construction of different formulations of diophantine triples with the property \(D(n)\) for any arbitrary integer \(n\) [1] and also, for any linear polynomials in \(n\). In this context, one may refer [2-13] for an extensive review of various problems on diophantine triples.
This paper concerns with the construction of sequences of diophantine 3-tuples \((a, b, c)\) from the pair of integers \(\{u, v\}\) such that the product of any two elements of the set added by
\[
D(a^2k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2) = \text{perfect square}
\]
is a perfect square. This paper is the generalization of [13].

**METHOD OF ANALYSIS**

Let \(u, v\) be any two given non-zero integers. For convenience and clear understanding, take
\[
a = u, c_0 = v
\]
It is observed that
\[
a c_0 + a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2 = (\alpha k + s - w)^2
\]
Therefore, the pair \((a, c_0)\) represents diophantine 2-tuple with the property
\[
D(a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2)
\]
Let \(c_1\) be any non-zero polynomial such that
\[
a c_1 + a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2 = p^2 \tag{1}
\]
\[
c_0c_1 + a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2 = q^2 \tag{2}
\]
Eliminating \(c_1\) between (1) and (2), we have
\[
c_0 p^2 - a q^2 = (c_0 - a) (a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2) \tag{3}
\]
Introducing the linear transformations
\[
p = X + aT, \quad q = X + c_0 T \tag{4}
\]
in (3) and simplifying we get
\[
X^2 = ac_0 T^2 + (a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2)
\]
which is satisfied by \(T = 1\), \(X = \alpha k + s - w\)
In view of (4) and (1), it is seen that
\[
c_1 = 2(\alpha k + s) + u + v - 2w
\]
Note that \((a, c_0, c_1)\) represents diophantine 3-tuple with property
\[
D(a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2)
\]
Taking \((a, c_1)\) and employing the above procedure, it is seen that the triple \((a, c_1, c_2)\) where
\[
c_2 = 4(\alpha k + s) + 4u + v - 4w
\]
exhibits diophantine 3-tuple with property \(D(a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2)\)
Taking \((a, c_2)\) and employing the above procedure, it is seen that the triple \((a, c_2, c_3)\) where
\[
c_3 = 6(\alpha k + s) + 9u + v - 6w
\]
exhibits diophantine 3-tuple with property \(D(a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2)\)
Taking \((a, c_3)\) and employing the above procedure, it is seen that the triple \((a, c_3, c_4)\) where
\[
c_4 = 8(\alpha k + s) + 16u + v - 8w
\]
exhibits diophantine 3-tuple with property \(D(a^2 k^2 + 2\alpha k(s - w) + s^2 - 2sw - uv + w^2)\)
The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by \((a, c_{\varrho - 1}, c_{\varrho})\) where
\[
c_{\varrho - 1} = 2(\varrho - 1)(\alpha k + s) + (\varrho - 1)^2 u + v - 2(\varrho - 1)w, \quad \varrho = 1, 2, 3, \ldots\]
A few numerical examples are presented in Table below:

<table>
<thead>
<tr>
<th>α</th>
<th>k</th>
<th>s</th>
<th>w</th>
<th>u</th>
<th>v</th>
<th>(a,c₁,c₂₁)</th>
<th>(a,c₁,c₂)</th>
<th>(a,c₂,c₃)</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>(2, 3, 7)</td>
<td>(2, 7, 15)</td>
<td>(2, 15, 27)</td>
<td>D(−5)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>(3, 2, 7)</td>
<td>(3, 7, 18)</td>
<td>(3, 18, 35)</td>
<td>D(−5)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>5</td>
<td>(−2, 5, 7)</td>
<td>(−2, 7, 5)</td>
<td>(−2, 5, −1)</td>
<td>D(14)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>(2, 2, 6)</td>
<td>(2, 6, 14)</td>
<td>(2, 14, 26)</td>
<td>D(−3)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2n −1</td>
<td>2n</td>
<td>2n + 1</td>
<td>(2n, 2n + 1, 8n + 3)</td>
<td>(2n, 8n + 3, 18n + 5)</td>
<td>(2n, 18n + 5, 32n + 7)</td>
<td>D(2n + 1)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2n</td>
<td>2n</td>
<td>2n + 1</td>
<td>(2n, 2n + 1, 8n + 1)</td>
<td>(2n, 8n + 1, 18n + 1)</td>
<td>(2n, 18n + 1, 32n + 1)</td>
<td>D(−2n)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2n + 1</td>
<td>2n</td>
<td>2n −1</td>
<td>(2n, 2n −1, 8n −3)</td>
<td>(2n, 8n −3, 18n −5)</td>
<td>(2n, 18n −5, 32n −7)</td>
<td>D(−2n + 1)</td>
</tr>
<tr>
<td>1</td>
<td>k</td>
<td>s</td>
<td>-1</td>
<td>k</td>
<td>k+3</td>
<td>(k, k + 3, 4k + 2s + 5)</td>
<td>(k, 4k + 2s + 5, 9k + 4s + 7)</td>
<td>(k, 9k + 4s + 7, 16k + 6s + 9)</td>
<td>D(2s + 1)k + 1</td>
</tr>
<tr>
<td>1</td>
<td>k</td>
<td>s</td>
<td>(1 −β)s</td>
<td>k</td>
<td>k+s</td>
<td>(k, k + s, 4k + (2β + 1)s)</td>
<td>(k, 4k + (2β + 1)s, 9k + (4β + 1)s)</td>
<td>(k, 9k + (4β + 1)s, 16k + (6β + 1)s)</td>
<td>D(2β −1)k + 1s²</td>
</tr>
<tr>
<td>1</td>
<td>k</td>
<td>2</td>
<td>-2</td>
<td>6</td>
<td>3</td>
<td>(6, 3, 2k + 17)</td>
<td>(6,4k + 17, 4k + 43)</td>
<td>(6,4k + 43, 6k + 81)</td>
<td>D(k² + 8k −2)</td>
</tr>
</tbody>
</table>

**CONCLUSION**

The researchers may attempt for the formulation of other sequences of diophantine 3-tuples such that the product of any two elements of the set added by a polynomial with integer coefficient is a perfect square.

**REFERENCES**

9. Gopalan, M.A and Geetha.V (December-January 2015), Formation of Diophantine Triples for Polygonal Numbers \(t_{16,n}\) to \(t_{25,n}\) and Centered Polygonal Numbers \(ct_{16,n}\) to \(ct_{25,n}\). IJITR, volume 3, Issue 1, 1837-1841.


13. S.Vidhyalakshmi, T.Mahalakshmi and M.A.Gopalan (July 2020), Formulation of Sequences of Diophantine 3-Tuples Through the Pair (3,6), EPRA-IJMR, Volume 6, Issue 7, 241-257.